Renormalizing a Becchi-Rouet-Stora-Tyutin-invariant composite operator of mass dimension 2 in Yang-Mills theory

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We discuss the renormalization of a Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST invariant composite operator of mass dimension 2 in Yang-Mills theory with general BRST and anti-BRST invariant gauge-fixing terms of Lorentz type. The interest of this study stems from a recent claim that the nonvanishing vacuum condensate of the composite operator in question can be an origin of mass gap and quark confinement in any manifestly covariant gauge, as proposed by one of the authors. First, we obtain the renormalization group flow of the Yang-Mills theory. Next, we show the multiplicative renormalizability of the composite operator and that the BRST and anti-BRST invariance of the bare composite operator is preserved under the renormalization. Third, we perform the operator product expansion of the gluon and ghost propagators and obtain the Wilson coefficient corresponding to the vacuum condensate of mass dimension 2. Finally, we discuss the connection of this work with previous works and argue the physical implications of the obtained results.

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I. INTRODUCTION

It is still a challenging and unsolved problem to prove quark confinement in the framework of quantum chromodynamics (QCD). A very beginning question in deriving quark confinement is in what sense is the quark confined? A simple criterion of quark confinement which has been widely used so far is the area law decay of the Wilson loop (defined by the vacuum expectation value of the Wilson loop operator). The area law implies the presence of a linear piece σr proportional to the interquark distance r in the static interquark potential $V(r)$. The dual superconductivity of QCD vacuum [1] is one of the most promising mechanisms of quark confinement compatible with this picture. However, it is well known that this criterion is not so useful in the presence of dynamical matter, since the interquark force is screened by a quark-antiquark pair created from the vacuum and the linear piece no longer appears in the potential.

In a previous paper [2], one of the authors (K.-I. K.) proposed a nonvanishing vacuum condensate $\langle \mathcal{O} \rangle$ of mass dimension 2 as the origin of mass gap and quark confinement in Yang-Mills theory. The proposed composite operator of mass dimension 2 is given by

$$\mathcal{O} := \frac{1}{\Omega^{(D)}} \int d^D x \ \mathrm{tr}\left[\frac{1}{2}\mathcal{A}_\mu(x)\cdot\mathcal{A}_\mu(x) + \alpha i \bar{\mathcal{C}}(x)\cdot\mathcal{C}(x)\right], \tag{1.1}$$

where $\mathcal{A}_\mu$ is the gauge field, $\mathcal{C}(\bar{\mathcal{C}})$ is the ghost (antighost) field, and $\Omega^{(D)}$ denotes the volume of the D-dimensional spacetime. It has been shown [2] that the composite operator $\mathcal{O}$ is invariant under the Becchi-Rouet-Stora-Tyutin (BRST) [3] and anti-BRST [4] transformations in the manifestly Lorentz covariant gauge, especially in the most general[1] Lorentz gauge [5–10] and the maximal Abelian (MA) gauge [11–18]. In Eq. (1.1), the trace is taken over the broken generators of the Lie algebra $\mathcal{G}$ of the original group G when the original gauge group G is broken to H by a local gauge-fixing condition chosen, i.e., G itself in the Lorentz gauge and G/H in the MA gauge corresponding to the maximal torus group H of G. Especially, in the limit $\alpha \to 0$ (which we call the Landau gauge), the composite operator reduces to $\mathcal{O} = (\Omega^{(D)})^{-1} \int d^D x \ \mathrm{tr}[1/2 \mathcal{A}_\mu(x)\cdot\mathcal{A}_\mu(x)]$ and hence becomes gauge invariant, since the contributions from the ghost and antighost disappear. The vacuum condensate includes the ghost condensation proposed in the MA gauge [19,20] and reduces to the gluon condensation recently proposed by several authors [21–24], see also Refs. [25,26].

The physical implication of the existence of such a condensate $\langle \mathcal{O} \rangle$ has been argued based on the operator product expansion (OPE) of the gluon and ghost propagators (two-point functions) and the vertex function (three-point function) [2,21,24]. However, the actual calculation has been performed within the tree level.

In order for such a proposal to be meaningful, it is very indispensable to show that the whole strategy to derive quark confinement based on the novel vacuum condensate survives the renormalization. In view of this, we focus on the renormalization of the composite operator (1.1). The main purpose of this paper is to examine whether or not the composite

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[1]The precise definition of "the most general" is stated later in the text. Roughly speaking, the most general Lorentz gauge is obtained by imposing both the BRST and anti-BRST invariance for the gauge fixing term which corresponds to the Lorentz gauge $\partial^\mu \mathcal{A}_\mu(x) = 0$. The resulting gauge-fixing term has two parameters. The conventional Lorentz gauge is obtained as a special choice of the parameters.

operator in the integrand of $\mathcal{O}$ is renormalizable. In addition, we must clarify the meaning of the BRST and anti-BRST symmetry in the renormalized theory. We examine whether or not the renormalized composite operator $\mathcal{O}^R$ is invariant under the renormalized BRST and anti-BRST transformations. If this is the case, the proposed composite operator of mass dimension 2 and the corresponding vacuum condensate can have a definite physical meaning. The analysis of this paper is restricted to the most general Lorentz gauge fixing, since the analysis of the MA gauge is more involved and hence the result is to be reported in a separate paper [27].

In the most general Lorentz gauge, the multiplicative renormalizability of Yang-Mills theory has been worked out by Baulieu and Thierry-Mieg [8] by making use of Slavnov-Taylor identities characterizing the BRST and anti-BRST invariance of the theory (see, e.g., Refs. [28–34]). In the course of renormalizing the composite operator, however, there is a subtle problem of the operator mixing. In order to discuss the renormalization of a composite operator, we must take into account all the contributions coming from all the other composite operators of the same mass dimension and the same symmetry property. In the OPE, the Wilson coefficient corresponding to an arbitrary vacuum condensate can be calculated by perturbation theory. In the usual Lorentz gauge, the Wilson coefficient associated with the ghost condensate $\langle \bar{\mathcal{C}} \cdot \mathcal{C} \rangle$ in the OPE of the propagator vanishes identically due to a special property of the three-point gluon-ghost-antighost vertex as pointed out in Ref. [35]. In the most general Lorentz gauge [8,9], however, we show in this paper that operator mixing between two composite operators $1/2\mathcal{A}_\mu \cdot \mathcal{A}_\mu$ and $i\bar{\mathcal{C}} \cdot \mathcal{C}$ of mass dimension 2 does exist in general due to the presence of four-ghost interaction (except for the case which is reduced to the conventional Lorentz gauge). We explicitly calculate the matrix of renormalization factors of the composite operator in the one-loop level.

For the Landau gauge, the vacuum condensate of mass dimension 2 in Yang-Mills theory is nothing but the gluon pair condensation. A possibility of gluon pair condensation was already suggested from the existence of the tachyon pole in the two gluon channel by approximately solving the Bethe-Salpeter equation; see, e.g., Refs. [37] and [38]. A gluon pair can be identified as a Cooper pair which is a bound state caused by the attractive force. Hence the gluon condensation is regarded as the Bose condensation of the gluon with spin 1. A remarkable point of our treatment that is different than the previous one is the retention of the manifest Lorentz covariance and gauge (or BRST and anti-BRST) invariance. Hence the introduction of the ghost field is indispensable in this approach. It is important to clarify how the inclusion of the ghost influences the dynamics of a gluon to recover the gauge invariance. This paper is a preliminary work toward the complete understanding of this problem.

Another purpose of this paper is to point out that the composite operator discussed above has an analogue in the Abelian gauge theory, especially, quantum electrodynamics (QED). This suggests that a confinement phase can exist even in QED, probably in the strong coupling region [39–42]. In QED, the vacuum condensate in question is reduced in the Landau gauge to photon pairing. Photon pairing has also been suggested long ago from the solution of the Cooper equation, see Refs. [43,44]. From quite a different viewpoint, one of the authors [36] discussed the existence of a confinement phase in QED based on the total QED Lagrangian with the BRST and anti-BRST invariant gauge-fixing terms which is identical to the usual Lagrangian in the Lorentz gauge up to a total derivative term. An advantage of rewriting the gauge-fixing part of the Lagrangian into the BRST and anti-BRST exact form is that the hidden supersymmetry becomes manifest and that the gauge-fixing part in four spacetime dimensions is reduced to the $O(2)$ nonlinear sigma model in two spacetime dimensions owing to Parisi-Sourlas dimensional reduction.[2] In view of this, the ghost is indispensable in this approach even for Abelian gauge theory where the ghost decouples and is usually considered to be unnecessary. In the analysis of quark confinement, it is most important to understand the origin of the scale or the mechanism of mass generation which was not so clear in previous treatments. The detailed analysis of this issue will be reported in a later paper.

This paper is organized as follows. In Sec. II, we summarize the BRST and anti-BRST transformations and their properties which are necessary in the following analyses. In Sec. III, we examine how the renormalization in QED is performed so as to preserve BRST and anti-BRST symmetry. This section is a preliminary step for dealing with non-Abelian gauge theory in the subsequent sections.

In Sec. IV, we consider the most general Lagrangian of Yang-Mills theory which has manifest Lorentz covariance, global gauge invariance, and BRST and anti-BRST symmetry. The gauge-fixing term contains two gauge-fixing parameters. We give the Feynman rules of this theory and calculate the renormalization constants in the one-loop level. Although some materials in this section are a reconfirmation of the results obtained by Baulieu and Thierry-Mieg [8], it is necessary to make this paper self-contained and to give basic ingredients in the subsequent sections.

In Sec. V, we obtain the renormalization group flow in the parameter space of the theory. To one-loop order, we specify the location of the fixed points and obtain the equation of the lines of connecting fixed points.

In Sec. VI, we discuss the main subject of this paper: the renormalization of the composite operator $\mathcal{O}$ of mass dimension 2. First, we show when the composite operator $\mathcal{O}$ is both BRST and anti-BRST invariant. Next, we evaluate the renormalization of $\mathcal{O}$ by taking into account the mixing of operators with the same mass dimensions and the same symmetry. To the best of our knowledge, the renormalization of the composite operator of mass dimension 2 has not been fully discussed except for a special case, i.e., the Landau gauge in conventional Lorentz gauge fixing [22].

In Sec. VII, we perform the operator product expansion of the gluon and ghost propagators and obtain the Wilson coef-

[2]This formulation has been applied to QED at finite temperature and a new confining phase is claimed to exist, see Ref. [45], and references therein.

ficient associated with the vacuum condensates in question. In the final section, we give the conclusions of this paper and discuss future directions of our research. In the Appendix, we give some of the calculations omitted in the text.

II. BRST AND ANTI-BRST TRANSFORMATIONS

We consider general non-Abelian gauge theory with a gauge group G. In the following we use the notation

$$F\cdot G := F^A G^A, \quad F^2 := F\cdot F, \quad (F\times G)^A := f^{ABC}F^B G^C, \tag{2.1}$$

where f^{ABC} are the structure constants of the Lie algebra $\mathcal{G}$ of the gauge group G.

For non-Abelian gauge theory, we define the BRST transformation by

$$\delta_B \mathcal{A}_\mu(x) = \mathcal{D}_\mu[\mathcal{A}]\mathcal{C}(x) := \partial_\mu \mathcal{C}(x) + g[\mathcal{A}_\mu(x)\times \mathcal{C}(x)], \tag{2.2a}$$

$$\delta_B \mathcal{C}(x) = -\frac{1}{2} g[\mathcal{C}(x)\times\mathcal{C}(x)], \tag{2.2b}$$

$$\delta_B \bar{\mathcal{C}}(x) = i\mathcal{B}(x), \tag{2.2c}$$

$$\delta_B \mathcal{B}(x) = 0, \tag{2.2d}$$

where $\mathcal{A}_\mu$, $\mathcal{B}$, $\mathcal{C}$, and $\bar{\mathcal{C}}$ are the non-Abelian gauge field, the Nakanishi-Lautrup (NL) auxiliary field, and the Faddeev-Popov (FP) ghost and antighost fields, respectively. Another BRST transformation, i.e., anti-BRST transformation [4], is defined by

$$\bar{\delta}_B \mathcal{A}_\mu(x) = \mathcal{D}_\mu[\mathcal{A}]\bar{\mathcal{C}}(x) := \partial_\mu \bar{\mathcal{C}}(x) + g[\mathcal{A}_\mu(x)\times \bar{\mathcal{C}}(x)], \tag{2.3a}$$

$$\bar{\delta}_B \bar{\mathcal{C}}(x) = -\frac{1}{2} g[\bar{\mathcal{C}}(x)\times\bar{\mathcal{C}}(x)], \tag{2.3b}$$

$$\bar{\delta}_B \mathcal{C}(x) = i\bar{\mathcal{B}}(x), \tag{2.3c}$$

$$\bar{\delta}_B \bar{\mathcal{B}}(x) = 0, \tag{2.3d}$$

where $\bar{\mathcal{B}}$ is defined by[3]

$$\bar{\mathcal{B}}(x) = -\mathcal{B}(x) + ig[\mathcal{C}(x)\times\bar{\mathcal{C}}(x)]. \tag{2.5}$$

The BRST and anti-BRST transformations are nilpotent and they anticommute:

$$\delta_B\delta_B \equiv 0, \quad \bar{\delta}_B\bar{\delta}_B = 0, \quad \delta_B\bar{\delta}_B + \bar{\delta}_B\delta_B \equiv 0. \tag{2.6}$$

For Abelian gauge theory, the BRST transformation reads

$$\delta_B a_\mu(x) = \partial_\mu C(x), \tag{2.7a}$$

$$\delta_B C(x) = 0, \tag{2.7b}$$

$$\delta_B \bar{C}(x) = iB(x), \tag{2.7c}$$

$$\delta_B B(x) = 0, \tag{2.7d}$$

where A_μ, B, C, and $\bar{C}$ are the Abelian gauge field, the NL auxiliary field, and the FP ghost and antighost fields, respectively. The anti-BRST transformation is reduced to

$$\bar{\delta}_B a_\mu(x) = \partial_\mu C(x), \tag{2.8a}$$

$$\bar{\delta}_B \bar{C}(x) = 0, \tag{2.8b}$$

$$\bar{\delta}_B C(x) = i\bar{B}(x), \tag{2.8c}$$

$$\bar{\delta}_B \bar{B}(x) = 0, \tag{2.8d}$$

where $\bar{B}$ is defined by

$$\bar{B}(x) = -B(x). \tag{2.9}$$

III. QED IN THE LORENTZ GAUGE

As a warming-up problem, we consider quantum electrodynamics. As is well known, the total Lagrangian of QED is given by

$$\mathcal{L}^{\text{tot}}_{\text{QED}} = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi a_\mu + \mathcal{L}_{\text{GF+FP}}, \tag{3.1}$$

with a gauge-fixing (GF) term plus a FP ghost term $\mathcal{L}_{\text{GF+FP}}$. The explicit form of the GF+FP term depends on the gauge chosen. In this paper we adopt the most familiar covariant gauge, i.e., the Lorentz gauge

$$\partial^\mu a_\mu = 0. \tag{3.2}$$

Therefore, the GF+FP term is given by

$$\mathcal{L}_{\text{GF+FP}} = i\delta_B\left(\bar{C}\partial^\mu a_\mu + \frac{\alpha}{2}\bar{C}B\right) = B\partial^\mu a_\mu + \frac{\alpha}{2}B^2 + i\bar{C}\partial^\mu\partial_\mu C. \tag{3.3}$$

Although the ghost and antighost fields are free and decouple from other fields, we have included them to study the relationship with the non-Abelian case which will be discussed in the next section.

As pointed out in Ref. [36], the GF+FP term (3.3) is rewritten into the BRST and anti-BRST exact form:

$$\mathcal{L}_{\text{GF+FP}} = i\delta_B\bar{\delta}_B\left(\frac{1}{2}a_\mu a^\mu + \frac{\alpha}{2}i\bar{C}C\right). \tag{3.4}$$

In fact, this is cast into the form

[3] The last transformation is equivalent to

$$\bar{\delta}_B\mathcal{B}(x) = -g\bar{\mathcal{C}}(x)\times\mathcal{B}(x). \tag{2.4}$$

$$\mathcal{L}_{\rm GF+FP} = i\delta_B\left((\bar{\delta}_B a^\mu)a_\mu - \frac{\alpha}{2} i\bar{C}\bar{\delta}_B C\right)$$
$$= i\delta_B\left(\partial^\mu \bar{C} a_\mu - \frac{\alpha}{2}\bar{C}B\right), \tag{3.5}$$

which agrees with Eq. (3.3) up to a total-derivative term.

If the NL field B is eliminated by performing the functional integration or by making use of the equation of motion, then we obtain

$$\mathcal{L}'_{\rm GF+FP} = -\frac{1}{2\alpha}(\partial^\mu a_\mu)^2 + i\bar{C}\partial^\mu\partial_\mu C. \tag{3.6}$$

The on-shell BRST transformation is given by

$$\delta_B a_\mu(x) = \partial_\mu C(x), \tag{3.7a}$$
$$\delta_B C(x) = 0, \tag{3.7b}$$
$$\delta_B \bar{C}(x) = -\frac{i}{\alpha}\partial^\mu a_\mu(x), \tag{3.7c}$$

while the on-shell anti-BRST transformation is

$$\bar{\delta}_B a_\mu(x) = \partial_\mu \bar{C}(x), \tag{3.8a}$$
$$\bar{\delta}_B \bar{C}(x) = 0, \tag{3.8b}$$
$$\bar{\delta}_B C(x) = +\frac{i}{\alpha}\partial^\mu a_\mu(x). \tag{3.8c}$$

The GF+FP Lagrangian $\mathcal{L}'_{\rm GF+FP}$ and the total Lagrangian $\mathcal{L}^{\rm tot}_{\rm QED}$ with $\mathcal{L}'_{\rm GF+FP}$ are separately invariant under the on-shell BRST and on-shell anti-BRST transformations. The nilpotency of the on-shell BRST and anti-BRST transformations is realized only when the equation of motion for the ghost and antighost fields is used, since

$$(\delta_B)^2 a_\mu(x) = 0, \tag{3.9a}$$
$$(\delta_B)^2 C(x) = 0, \tag{3.9b}$$
$$(\delta_B)^2 \bar{C}(x) = -\frac{i}{\alpha}\partial^\mu\partial_\mu C(x) \tag{3.9c}$$

and

$$(\bar{\delta}_B)^2 a_\mu(x) = 0, \tag{3.10a}$$
$$(\bar{\delta}_B)^2 C(x) = +\frac{i}{\alpha}\partial^\mu\partial_\mu \bar{C}(x), \tag{3.10b}$$
$$(\bar{\delta}_B)^2 \bar{C}(x) = 0. \tag{3.10c}$$

Moreover, we obtain a similar result for the anticommutability:

$$(\delta_B\bar{\delta}_B + \bar{\delta}_B\delta_B)a_\mu(x) = 0, \tag{3.11a}$$
$$(\delta_B\bar{\delta}_B + \bar{\delta}_B\delta_B)C(x) = -\frac{i}{\alpha}\partial^\mu\partial_\mu \bar{C}(x), \tag{3.11b}$$
$$(\delta_B\bar{\delta}_B + \bar{\delta}_B\delta_B)\bar{C}(x) = +\frac{i}{\alpha}\partial^\mu\partial_\mu \bar{C}(x). \tag{3.11c}$$

Now we define the composite operator $\mathcal{O}$ of mass dimension 2 as

$$\mathcal{O} := \frac{1}{\Omega^{(D)}}\int d^D x\, \mathcal{Q}(x),$$
$$\mathcal{Q}(x) := \frac{1}{2}a_\mu(x)a^\mu(x) + \alpha i\bar{C}(x)C(x). \tag{3.12}$$

This composite operator is BRST and anti-BRST invariant, since

$$\delta_B\mathcal{Q}(x) = \partial^\mu[a_\mu(x)C(x)], \quad \bar{\delta}_B\mathcal{Q}(x) = \partial^\mu[a_\mu(x)\bar{C}(x)]. \tag{3.13}$$

We consider the renormalization of the composite operator $\mathcal{Q}$. The Abelian case is very simple due to the trivial renormalization factors Z_{a^2}, $Z_{\bar{C}C}$ for the composite fields $1/2a^\mu a_\mu$ and $i\bar{C}C$. Therefore, we only have to take into account the renormalization factor of the fundamental field, $a_\mu, C, \bar{C}$, and the gauge-fixing parameter α. QED is known to be multiplicatively renormalizable in the sense that the divergences are absorbed by introducing the renormalization factors in the following way:

$$\psi = Z_2^{1/2}\psi^R, \tag{3.14}$$
$$a_\mu = Z_3^{1/2}a_\mu^R, \tag{3.15}$$
$$C = Z_C C^R, \quad \bar{C} = Z_{\bar{C}}\bar{C}^R, \tag{3.16}$$
$$(B = Z_3^{-1/2}B^R), \tag{3.17}$$
$$m = Z_m Z_2^{-1}m^R, \tag{3.18}$$
$$\alpha = Z_\alpha \alpha^R, \tag{3.19}$$
$$e = Z_1 Z_2^{-1}Z_3^{-1/2}e^R. \tag{3.20}$$

The renormalization factors are not independent to each other. In fact, the coupling constant is renormalized as

$$e = Z_3^{-1/2}e^R, \tag{3.21}$$

as a consequence of the Ward relation

$$Z_1 = Z_2. \tag{3.22}$$

Moreover, the Ward-Takahashi identity yields

$$Z_\alpha = Z_3. \tag{3.23}$$

The result of perturbative renormalization in QED is well known and can be seen in the textbooks. The result

$$Z_C = Z_{\bar{C}} = 1 \tag{3.24}$$

means that both the ghost and antighost are free and receive no renormalization in the perturbation theory (this is not so in the non-Abelian case). Consequently, we arrive at the result that the composite operator is renormalized as

$$\mathcal{Q} = Z_3 \mathcal{Q}^R, \quad \mathcal{Q}^R := \frac{1}{2} a_\mu^R(x) a^{\mu R}(x) + \alpha^R i \bar{C}^R(x) C^R(x). \tag{3.25}$$

Therefore, the BRST invariant combination of two composite operators with mass dimension 2 is preserved under the renormalization.

In view of the above results, the renormalized BRST transformation is defined by

$$\delta_B^R = Z_3^{1/2} \delta_B, \quad \bar{\delta}_B^R = Z_3^{1/2} \bar{\delta}_B. \tag{3.26}$$

This is shown as follows. The Noether current of the BRST symmetry is obtained as

$$J_B^\mu = B \partial^\mu C - \partial^\mu B C - \partial_\nu (f^{\mu\nu} C). \tag{3.27}$$

The Noether charge, i.e., the BRST charge Q_B as the generator of the BRST transformation

$$[i\lambda Q_B, \Phi(x)] = \lambda \delta_B \Phi(x), \tag{3.28}$$

is given by

$$Q_B = \int d^3x J_B^0 = \int d^3x [B \partial^0 C - \partial^0 B C]. \tag{3.29}$$

In a similar way, the anti-BRST charge $\bar{Q}_B$ can also be defined as the Noether charge for the anti-BRST transformation. Therefore we can define the renormalized BRST charge Q_B^R as

$$Q_B^R = Z_3^{1/2} Q_B = \int d^3x [B^R \partial^0 C^R - \partial^0 B^R C^R]. \tag{3.30}$$

This ensures the renormalization of the BRST transformation (3.26). The renormalized BRST transformation for the renormalized field has the same form as the bare BRST transformation for the bare field. Thus, the composite operator $\mathcal{Q}$ is a BRST invariant and multiplicatively renormalizable operator for arbitrary gauge parameter α. The renormalized GF +FP term has the same form as the bare one:

$$\mathcal{L}_{\text{GF+FP}} = i \delta_B^R \bar{\delta}_B^R \left(\frac{1}{2} a_\mu^R a^{\mu R} + \frac{\alpha^R}{2} i \bar{C}^R C^R \right). \tag{3.31}$$

IV. YANG-MILLS THEORY IN THE MOST GENERAL LORENTZ GAUGE

A. Lagrangian

We consider the most general quantum Lagrangian density that is a local function of the fields $\mathcal{A}_\mu^A$, $\mathcal{B}^A$, $\mathcal{C}^A$, $\bar{\mathcal{C}}^A$ and satisfies the following conditions. The Lagrangian is (1) of mass dimension 4, (2) Lorentz invariant, (3a) BRST invariant, (3b) anti-BRST invariant, (4) Hermitian, (5) of zero ghost number, (6) global gauge invariant, and the theory with this Lagrangian is (7) (multiplicative) renormalizable. Here it is implicitly assumed that the Lagrangian is written as the polynomial of the fields, and that there are no higher derivative terms, since there is no intrinsic mass scale in Yang-Mills theory. It should be remarked that we have imposed BRST and anti-BRST invariance instead of gauge invariance (we do not require gauge invariance for the Lagrangian). Such a Lagrangian was given by Baulieu and Thierry-Mieg [8,9] as

$$\mathcal{L}_{YM}^{\text{tot}} = -\frac{1}{4} \alpha_1 \mathcal{F}_{\mu\nu} \cdot \mathcal{F}^{\mu\nu} + \alpha_2 \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\mu\nu} \cdot \mathcal{F}^{\rho\sigma} + i \delta_B \bar{\delta}_B (\alpha_3 \mathcal{A}_\mu \cdot \mathcal{A}^\mu + \alpha_4 \mathcal{C} \cdot \bar{\mathcal{C}}) + \frac{\alpha'}{2} \mathcal{B} \cdot \mathcal{B}, \tag{4.1}$$

where α_i ($i=1,2,3,4$) is an arbitrary constant and δ_B and $\bar{\delta}_B$ are the BRST and anti-BRST transformations. The first term is the Yang-Mills Lagrangian and the second term is the topological term which is not discussed in this paper and omitted hereafter. The first and second terms are gauge invariant. On the other hand, the third and the fourth terms are identified as the GF and FP terms, since they break the gauge invariance of the Lagrangian. After rescaling the parameters and the field redefinitions, we can cast the total Lagrangian of the Yang-Mills theory into the form

$$\mathcal{L}_{\text{YM}}^{\text{tot}} = -\frac{1}{4} \mathcal{F}_{\mu\nu} \cdot \mathcal{F}^{\mu\nu} + \mathcal{L}_{\text{GF+FP}}, \tag{4.2}$$

with the GF+FP term [8–10]

$$\mathcal{L}_{\text{GF+FP}} = i \delta_B \bar{\delta}_B \left(\frac{1}{2} \mathcal{A}_\mu \cdot \mathcal{A}^\mu - \frac{\alpha}{2} i \mathcal{C} \cdot \bar{\mathcal{C}} \right) + \frac{\alpha'}{2} \mathcal{B} \cdot \mathcal{B} \tag{4.3}$$

$$= -i \delta_B \left(-\partial_\mu \bar{\mathcal{C}} \cdot \mathcal{A}^\mu + \frac{\alpha}{2} \bar{\mathcal{C}} \cdot \mathcal{B} - \frac{i}{4} \alpha g \bar{\mathcal{C}} \cdot (\bar{\mathcal{C}} \times \mathcal{C}) \right) + \frac{\alpha'}{2} \mathcal{B} \cdot \mathcal{B}. \tag{4.4}$$

The final term is allowed for the renormalizability of the total Lagrangian and is written in either a BRST exact or anti-BRST exact form

$$\mathcal{B} \cdot \mathcal{B} = -i \delta_B (\bar{\mathcal{C}} \cdot \mathcal{B}) = i \bar{\delta}_B (\mathcal{C} \cdot \mathcal{B}). \tag{4.5}$$

However, the GF+FP term (4.4) is simultaneously BRST and anti-BRST exact, i.e., $\delta_B \bar{\delta}_B(*)$, only if $\alpha'=0$. If we impose one more condition, e.g., the FP ghost conjugation invariance

$$\mathcal{C}^A \to \pm \bar{\mathcal{C}}^A, \quad \bar{\mathcal{C}}^A \to \mp \mathcal{C}^A, \quad \mathcal{B}^A \to -\bar{\mathcal{B}}^A, \quad \bar{\mathcal{B}}^A \to -\mathcal{B}^A$$
$$(\mathcal{A}_\mu^A \to \mathcal{A}_\mu^A), \tag{4.6}$$

the second term of Eq. (4.4) is excluded, namely, only the choice $\alpha'=0$ is allowed.

By performing the BRST and anti-BRST transformations, we obtain

$$\mathcal{L}_{\mathrm{GF+FP}} = \frac{\alpha+\alpha'}{2}\mathcal{B}\cdot\mathcal{B} - \frac{\alpha}{2} ig(\mathcal{C}\times\bar{\mathcal{C}})\cdot\mathcal{B} + \mathcal{B}\cdot\partial_\mu\mathcal{A}^\mu + i\bar{\mathcal{C}}\cdot\partial_\mu\mathcal{D}^\mu[\mathcal{A}]\mathcal{C} + \frac{\alpha}{8}g^2(\bar{\mathcal{C}}\times\bar{\mathcal{C}})\cdot(\mathcal{C}\times\mathcal{C}), \tag{4.7}$$

$$= \frac{\alpha+\alpha'}{2}\mathcal{B}\cdot\mathcal{B} - \frac{\alpha}{2} ig(\mathcal{C}\times\bar{\mathcal{C}})\cdot\mathcal{B} + \mathcal{B}\cdot\partial_\mu\mathcal{A}^\mu + i\bar{\mathcal{C}}\cdot\partial_\mu\mathcal{D}^\mu[\mathcal{A}]\mathcal{C} + \frac{\alpha}{4}g^2(i\mathcal{C}\times\bar{\mathcal{C}})\cdot(i\mathcal{C}\times\bar{\mathcal{C}}). \tag{4.8}$$

The GF+FP term includes the ghost self-interaction where the strength is proportional to the parameter α.

When $\alpha=0$, this theory reduces to usual Yang-Mills theory in the Lorentz-type gauge fixing with the gauge-fixing parameter α':

$$\mathcal{L}_{\mathrm{GF+FP}} = \frac{\alpha'}{2}\mathcal{B}\cdot\mathcal{B} + \mathcal{B}\cdot\partial_\mu A^\mu + i\bar{\mathcal{C}}\cdot\partial_\mu\mathcal{D}^\mu[\mathcal{A}]\mathcal{C}. \tag{4.9}$$

This is consistent with the FP prescription.

When $\alpha\neq 0$, there exists a quartic ghost interaction which cannot be implemented by the usual FP prescription. Therefore we must go beyond the FP prescription. The GF+FP term is further rewritten as

$$\mathcal{L}_{\mathrm{GF+FP}} = -\frac{1}{2\lambda}(\partial^\mu\mathcal{A}_\mu)^2 + (1-\xi)i\bar{\mathcal{C}}\cdot\partial_\mu\mathcal{D}^\mu[\mathcal{A}]\mathcal{C} + \xi i\bar{\mathcal{C}}\cdot\mathcal{D}^\mu[\mathcal{A}]\partial_\mu\mathcal{C} + \frac{1}{2}\lambda\xi(1-\xi)g^2(i\mathcal{C}\times\bar{\mathcal{C}})\cdot(i\mathcal{C}\times\bar{\mathcal{C}}) + \frac{\lambda}{2}[\mathcal{B}+\lambda^{-1}\partial^\mu\mathcal{A}_\mu - \xi ig(\mathcal{C}\times\bar{\mathcal{C}})]^2, \tag{4.10}$$

$$= -\frac{1}{2\lambda}(\partial^\mu\mathcal{A}_\mu)^2 + i\bar{\mathcal{C}}\cdot\partial_\mu\partial^\mu\mathcal{C} - (1-\xi)gi\mathcal{A}^\mu\cdot(\partial_\mu\bar{\mathcal{C}}\mathcal{C}) + \xi gi\mathcal{A}^\mu\cdot(\bar{\mathcal{C}}\times\partial_\mu\mathcal{C}) + \frac{1}{2}\lambda\xi(1-\xi)g^2(i\mathcal{C}\times\bar{\mathcal{C}})\cdot(i\mathcal{C}\times\bar{\mathcal{C}}) + \frac{\lambda}{2}[\mathcal{B}+\lambda^{-1}\partial^\mu\mathcal{A}_\mu - \xi ig(\mathcal{C}\times\bar{\mathcal{C}})]^2, \tag{4.11}$$

where we have defined the two parameters[4]

$$\lambda := \alpha+\alpha', \quad \xi := \frac{\alpha/2}{\alpha+\alpha'} = \frac{\alpha}{2\lambda}. \tag{4.12}$$

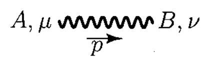
In this form, it is easy to eliminate the Nakanishi-Lautrup field $\mathcal{B}$. We call the gauge (4.11) the most general Lorentz gauge hereafter.

B. Feynman rules

We obtain the following Feynman rules for the Yang-Mills theory of the Lagrangian (4.2) with Eq. (4.11) where the NL field is eliminated.

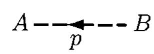
1. Propagators

(a) Gluon propagator:



$$= iD_{\mu\nu}^{AB} = -\frac{i}{p^2}\left[g_{\mu\nu} - (1-\lambda)\frac{p_\mu p_\nu}{p^2}\right]\delta^{AB}. \tag{4.13}$$

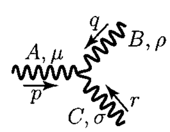
(b) Ghost propagator:



$$= iG^{AB} = -\frac{1}{p^2}\delta^{AB}. \tag{4.14}$$

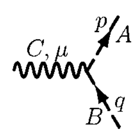
2. Three-point vertices

(c) Three-gluon vertex:



$$= gf^{ABC}[(q-r)_\mu g_{\rho\sigma} + (r-p)_\rho g_{\sigma\mu} + (p-q)_\sigma g_{\mu\rho}]. \tag{4.15}$$

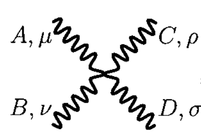
(d) Gluon-ghost-antighost vertex:



$$= igf^{ABC}[\xi(p-q)-p]^\mu. \tag{4.16}$$

3. Four-point vertices

(e) Four-gluon vertex:

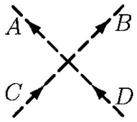


$$= -i2g^2\left(f^{EAB}f^{ECD}I_{\mu\nu,\rho\sigma} + f^{EAC}f^{EBD}I_{\mu\rho,\nu\sigma} + f^{EAD}f^{EBC}I_{\mu\sigma,\nu\rho}\right), \tag{4.17}$$

[4]The parameters α, α', λ, ξ in this paper correspond, respectively, to λ_c, λ_b, λ, α in Ref. [9] and a, a', λ, $\alpha/2$ in Ref. [8].

where $I_{\mu\nu,\rho\sigma} := (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})/2$.

(f) Four-ghost vertex:



$$= -i\lambda\xi(1-\xi)g^2\left(f^{ACE}f^{BDE} - f^{ADE}f^{BCE}\right). \tag{4.18}$$

C. Multiplicative renormalization

It has been proved by Baulieu and Thierry-Mieg [8] based on mathematical induction that the Yang-Mills theory in the most general Lorentz gauge (4.11) is multiplicatively renormalizable. We introduce the renormalization constant (or renormalization factor) for the field

$$\mathcal{A}_\mu = Z_A^{1/2}\mathcal{A}_\mu^R, \quad \mathcal{C} = Z_C^{1/2}\mathcal{C}^R, \quad \bar{\mathcal{C}} = Z_C^{1/2}\bar{\mathcal{C}}^R,$$

$$\mathcal{B} = Z_B^{1/2}\mathcal{B}^R = Z_C Z_A^{-1/2}\mathcal{B}^R \tag{4.19}$$

and for the parameters

$$\lambda = Z_\lambda \lambda_R, \quad \xi = Z_\xi \xi_R, \quad g = Z_g g_R. \tag{4.20}$$

By substituting Eqs. (4.19) and (4.20) into the bare Lagrangian, we obtain the total Lagrangian written in terms of the renormalized fields, renormalized parameters, and the renormalization factors:

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}} = & -\frac{1}{4}Z_A(\partial_\mu\mathcal{A}_\nu^R - \partial_\nu\mathcal{A}_\nu^R + Z_g Z_A^{1/2} g_R \mathcal{A}_\mu^R \times \mathcal{A}_\nu^R)^2 \\ & -\frac{1}{2\lambda_R}Z_A Z_\lambda^{-1}(\partial^\mu\mathcal{A}_\mu^R)^2 + iZ_C\bar{\mathcal{C}}^R\cdot\partial_\mu\partial^\mu\mathcal{C}^R \\ & -(1-Z_\xi\xi_R)Z_g Z_A^{1/2}Z_C g_R i\mathcal{A}^{\mu R}\cdot(\partial_\mu\bar{\mathcal{C}}^R\times\mathcal{C}^R) \\ & +Z_\xi Z_g Z_A^{1/2}Z_C\xi_R g_R i\mathcal{A}^{\mu R}\cdot(\bar{\mathcal{C}}^R\times\partial_\mu\mathcal{C}^R) \\ & +\frac{1}{2}Z_\lambda Z_\xi Z_g^2 Z_C^2\lambda_R\xi_R(1-Z_\xi\xi_R)g_R^2(i\mathcal{C}^R\times\bar{\mathcal{C}}^R)\cdot(i\mathcal{C}^R \\ & \times\bar{\mathcal{C}}^R) + \frac{\lambda_R}{2}Z_\lambda(Z_C Z_A^{-1/2}\mathcal{B}^R + Z_\lambda^{-1}Z_A^{1/2}\lambda_R^{-1}\partial^\mu\mathcal{A}_\mu^R \\ & -Z_\xi Z_g Z_C\xi_R i g_R\mathcal{C}^R\times\bar{\mathcal{C}}^R)^2.\end{aligned} \tag{4.21}$$

The total Lagrangian (4.21) is decomposed into a renormalization-factor independent part $\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}\,R}$ and the remaining part $\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot\,c}}$ as

$$\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}} = \mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}\,R} + \mathcal{L}_{\mathrm{YM}}^{\mathrm{tot\,c}}, \tag{4.22a}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}\,R} := & -\frac{1}{4}(\partial_\mu\mathcal{A}_\nu^R - \partial_\nu\mathcal{A}_\mu^R + g_R\mathcal{A}_\mu^R\times\mathcal{A}_\nu^R)^2 - \frac{1}{2\lambda_R}(\partial^\mu\mathcal{A}_\mu^R)^2 \\ & + i\bar{\mathcal{C}}^R\cdot\partial_\mu\partial^\mu\mathcal{C}^R - (1-\xi_R)g_R i\mathcal{A}^{\mu R}\cdot(\partial_\mu\bar{\mathcal{C}}^R\times\mathcal{C}^R) \\ & + \xi_R g_R i\mathcal{A}^{\mu R}\cdot(\bar{\mathcal{C}}^R\times\partial_\mu\mathcal{C}^R) \\ & + \frac{1}{2}\lambda_R\xi_R(1-\xi_R) \\ & \times g_R^2(i\mathcal{C}^R\times\bar{\mathcal{C}}^R)\cdot(i\mathcal{C}^R\times\bar{\mathcal{C}}^R) \\ & + \frac{\lambda_R}{2}(\mathcal{B}^R + \lambda_R^{-1}\partial^\mu\mathcal{A}_\mu^R - \xi_R i g_R\mathcal{C}^R\times\bar{\mathcal{C}}^R)^2,\end{aligned} \tag{4.22b}$$

$$\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot\,c}} := (4.21) - (4.22\mathrm{b}). \tag{4.22c}$$

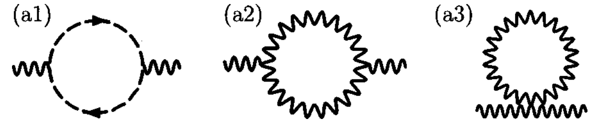


FIG. 1. Vacuum polarization of the gluon.

Here $\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}\,R}$ is obtained by setting all renormalization factors $Z \equiv 1$ in Eq. (4.21) and hence it is written in terms of the renormalized fields and renormalized parameters and has the same form as the bare Lagrangian $\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}}$, while $\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot\,c}}$ is the counterterm defined by the difference $\mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}} - \mathcal{L}_{\mathrm{YM}}^{\mathrm{tot}\,R}$.

1. Renormalization of two-point functions

First, we calculate the vacuum polarization function of the gluon. To the order g^2, there are three Feynman diagrams, see (a1), (a2), and (a3) in Fig. 1.

As a gauge-invariant regularization, we adopt the dimensional regularization. Then we obtain the following result ($\epsilon := 2 - D/2$):

$$\begin{aligned}(\mathrm{a1}) = & C_2(G)\delta^{AB}\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{i}{\epsilon}\left[\frac{1}{12}q^2 g_{\mu\nu}\right. \\ & \left.-\left\{\xi(1-\xi)-\frac{1}{6}\right\}q_\mu q_\nu\right],\end{aligned} \tag{4.23a}$$

$$\begin{aligned}(\mathrm{a2}) = & \frac{1}{2}C_2(G)\delta^{AB}\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{i}{\epsilon}\left\{\frac{19}{6}q^2 g_{\mu\nu}\right. \\ & \left.-\frac{11}{3}q_\mu q_\nu + (1-\lambda)(q^2 g_{\mu\nu} - q_\mu q_\nu)\right\},\end{aligned} \tag{4.23b}$$

$$(\mathrm{a3}) = 0, \tag{4.23c}$$

where $C_2 = C_2(G)$ is the quadratic Casimir operator in the adjoint representation of the gauge group G defined by $\delta^{AB}C_2(G) = f^{ACD}f^{BCD}$. Hence the counterterms δ_{T} and δ_{L} for the transverse and longitudinal part of the vacuum polarization tensor are determined so as to satisfy the relation

$$\begin{aligned}(\mathrm{a1}) + (\mathrm{a2}) + (\mathrm{a3}) - i\delta_{\mathrm{T}}(q^2 g_{\mu\nu} - q_\mu q_\nu)\delta^{AB} \\ - i\frac{\delta_{\mathrm{L}}}{\lambda}q_\mu q_\nu\delta^{AB} \equiv 0,\end{aligned} \tag{4.24}$$

which yields the result

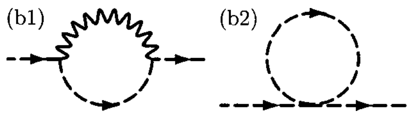


FIG. 2. Vacuum polarization of the ghost.

$$\delta_{\mathrm{T}}=\left(\frac{13}{6}-\frac{\lambda}{2}\right)\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon},$$

$$\delta_{\mathrm{L}}=-\lambda\xi(1-\xi)\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}. \tag{4.25}$$

On the other hand, the relationship

$$\delta_{\mathrm{T}}=Z_A-1=Z_A^{(1)}+\cdots,\quad \delta_{\mathrm{L}}=Z_AZ_\lambda^{-1}-1=Z_A^{(1)}-Z_\lambda^{(1)}+\cdots, \tag{4.26}$$

must hold for the multiplicative renormalizability where we have defined the renormalization factor Z order by order of the loop expansion $Z=1+Z^{(1)}+Z^{(2)}+\cdots$. Thus we obtain the renormalization factors

$$Z_A^{(1)}=\delta_{\mathrm{T}}=\left(\frac{13}{6}-\frac{\lambda}{2}\right)\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon} \tag{4.27}$$

and

$$Z_\lambda^{(1)}=\delta_{\mathrm{T}}-\delta_{\mathrm{L}}=\left[\left(\frac{13}{6}-\frac{\lambda}{2}\right)+\lambda\xi(1-\xi)\right]\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}. \tag{4.28}$$

Note that δ_T and hence Z_A is the same as in the FP case where the four ghost interaction does not exist. When $\xi\neq 0,1$, however, we find that $\delta_L\neq 0$ or equivalently $Z_A\neq Z_\lambda$. On the contrary to the FP case, the longitudinal part of the gluon propagator must be renormalized in this case.

Next, the vacuum polarization function of the ghost is calculated in a similar way. To order g^2, there are two Feynman diagrams, see (b1) and (b2) in Fig. 2. The explicit calculation shows that

$$(\mathrm{b1})=\left(\frac{1}{2}+\frac{1-\lambda}{4}\right)\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}p^2\delta^{AB}, \tag{4.29a}$$

$$(\mathrm{b2})=0. \tag{4.29b}$$

The counterterm δ_C is determined from

$$(\mathrm{b1})+(\mathrm{b2})-p^2\delta^{AB}\delta_C=0. \tag{4.30}$$

Hence the counterterm $\delta_C=Z_C-1=Z_C^{(1)}+\cdots$ is equal to the renormalization constant $Z_C^{(1)}$:

$$Z_C^{(1)}=\delta_C=\frac{3-\lambda}{4}\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}. \tag{4.31}$$

This is again the same as in the FP case.

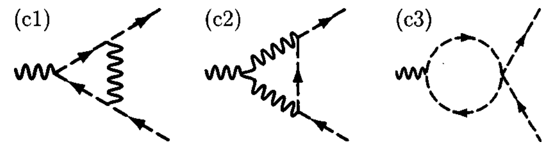
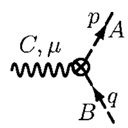


FIG. 3. Radiative corrections for the gluon-ghost-antighost vertex.

2. Renormalization of the three-point function

We consider the renormalization of three-point vertex. For example, the Feynman diagrams for the radiative correction of the gluon-ghost-antighost vertex to one-loop order is given in Fig. 3.

If we write the counterterm for the gluon-ghost-antighost vertex function as



$$=ig_{\mathrm{R}}f^{ABC}\left[\xi_{\mathrm{R}}\delta^1_{AC\bar{C}}(p-q)-\delta^2_{AC\bar{C}}p\right]_\mu, \tag{4.32}$$

we find the renormalization factors are related as

$$\delta^1_{AC\bar{C}}=Z_CZ_A^{1/2}Z_gZ_\xi-1=Z_C^{(1)}+\frac{1}{2}Z_A^{(1)}+Z_g^{(1)}+Z_\xi^{(1)}+\cdots, \tag{4.33}$$

$$\delta^2_{AC\bar{C}}=Z_CZ_A^{1/2}Z_g-1=Z_C^{(1)}+\frac{1}{2}Z_A^{(1)}+Z_g^{(1)}+\cdots. \tag{4.34}$$

At $p=q$, the respective diagram is calculated as

$$(\mathrm{c1})_{p=q}=-\frac{1}{2}C_2(G)f^{ABC}g^3\frac{i}{(4\pi)^2}\frac{1}{\epsilon}\frac{\lambda}{4}p^\mu, \tag{4.35a}$$

$$(\mathrm{c2})_{p=q}=-\frac{1}{2}C_2(G)f^{ABC}g^3\lambda\frac{i}{(4\pi)^2}\frac{1}{\epsilon}\frac{3}{4}p_\mu, \tag{4.35b}$$

$$(\mathrm{c3})_{p=q}=0. \tag{4.35c}$$

By substituting Eqs. (4.35a), (4.35b), and (4.35c) into

$$(\mathrm{c1})_{p=q}+(\mathrm{c2})_{p=q}+(\mathrm{c3})_{p=q}-igf^{ABC}\delta^2_{AC\bar{C}}p_\mu\equiv 0, \tag{4.36}$$

it follows that

$$\delta^2_{AC\bar{C}}=-\frac{1}{2}\lambda\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}. \tag{4.37}$$

Hence the renormalization factor is obtained as

$$Z_g^{(1)}=\delta^2_{AC\bar{C}}-Z_C^{(1)}-\frac{1}{2}Z_A^{(1)}=-\frac{11}{6}\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}. \tag{4.38}$$

At $p=0$, the respective diagram is calculated as

$$(\mathrm{c1})_{p=0} = -\frac{1}{2} C_2(G) f^{ABC} g^3 \xi \lambda \frac{i}{(4\pi)^2} \times \frac{1}{\epsilon}\left[(1-\xi)\left(\xi-\frac{1}{2}\right)+\frac{1}{4}\right] q^\mu, \quad (4.39a)$$

$$(\mathrm{c2})_{p=0} = -\frac{1}{2} C_2(G) f^{ABC} g^3 \lambda \xi \frac{i}{(4\pi)^2} \frac{1}{\epsilon} \frac{3}{4} q^\mu, \quad (4.39b)$$

$$(\mathrm{c3})_{p=0} = -\frac{1}{2} C_2(G) f^{ABC} g^3 \lambda \xi (1-\xi) \times \frac{i}{(4\pi)^2} \frac{1}{\epsilon}\left(\xi-\frac{1}{2}\right) q^\mu. \quad (4.39c)$$

By substituting Eqs. (4.39a), (4.39b), and (4.39c) into

$$(\mathrm{c1})_{p=0} + (\mathrm{c2})_{p=0} + (\mathrm{c3})_{p=0} - i g f^{ABC} \xi_R \delta^1_{AC\bar{C}} q_\mu \equiv 0, \quad (4.40)$$

it follows that

$$\delta^1_{AC\bar{C}} = \left[-\lambda(1-\xi)\left(\xi-\frac{1}{2}\right)-\frac{1}{2}\lambda\right] \frac{(g\mu^{-\epsilon})^2}{(4\pi)^2} \frac{C_2(G)}{\epsilon}. \quad (4.41)$$

Then we obtain

$$Z_\xi^{(1)} = \delta^1_{AC\bar{C}} - \delta^2_{AC\bar{C}} = \lambda(\xi-1)\left(\xi-\frac{1}{2}\right) \frac{(g\mu^{-\epsilon})^2}{(4\pi)^2} \frac{C_2(G)}{\epsilon}. \quad (4.42)$$

Accordingly, the renormalization constants of α and α' are obtained as

$$Z_\alpha^{(1)} = \left(\frac{13}{6}-\frac{\alpha}{4}\right) \frac{(g\mu^{-\epsilon})^2}{(4\pi)^2} \frac{C_2(G)}{\epsilon} \quad (4.43)$$

and

$$Z_{\alpha'}^{(1)} = \left(\frac{13}{6}-\frac{\alpha+\alpha'}{2}\right) \frac{(g\mu^{-\epsilon})^2}{(4\pi)^2} \frac{C_2(G)}{\epsilon}. \quad (4.44)$$

V. RENORMALIZATION GROUP FLOW AND FIXED POINTS

Using the above result, the renormalization group (RG) functions are obtained as follows. The β function is obtained as

$$\beta(g_R) := \mu \frac{\partial g_R}{\partial \mu} = -g_R \mu \frac{\partial}{\partial \mu} \ln Z_g \cong -g_R \mu \frac{\partial}{\partial \mu} Z_g^{(1)}. \quad (5.1)$$

It turns out that the β function does not depend on the gauge parameters λ and ξ:

$$\beta(g_R) := \mu \frac{\partial g_R}{\partial \mu} = -\frac{1}{16\pi^2} \frac{11}{3} C_2(G) g_R^3. \quad (5.2)$$

Similarly, we obtain the RG functions

$$\gamma_\xi := \mu \frac{\partial}{\partial \mu} \xi_R = 2\lambda_R \xi_R (\xi_R - 1)\left(\xi_R-\frac{1}{2}\right) \frac{C_2(G)}{(4\pi)^2} g_R^2 \quad (5.3)$$

and

$$\gamma_\lambda := \mu \frac{\partial}{\partial \mu} \lambda_R = 2\lambda_R\left[\frac{13}{6}-\frac{\lambda_R}{2}+\lambda_R \xi_R (1-\xi_R)\right] \frac{C_2(G)}{(4\pi)^2} g_R^2. \quad (5.4)$$

The RG flow in three-dimensional parameter space (ξ,λ,g) is determined by solving simultaneous differential equations

$$\mu \frac{\partial \xi}{\partial \mu} = 2\lambda \xi (\xi-1)\left(\xi-\frac{1}{2}\right) \frac{C_2(G) g^2}{(4\pi)^2}, \quad (5.5a)$$

$$\mu \frac{\partial \lambda}{\partial \mu} = 2\lambda\left[\frac{13}{6}-\frac{\lambda}{2}+\lambda \xi (1-\xi)\right] \frac{C_2(G) g^2}{(4\pi)^2}, \quad (5.5b)$$

$$\mu \frac{\partial g}{\partial \mu} = -\frac{11}{3} \frac{C_2(G) g^3}{(4\pi)^2}, \quad (5.5c)$$

where we have omitted the subscript R for the renormalized quantity.

As is well known, Eq. (5.5c) is solved exactly,

$$g^2(\mu) = \frac{g^2(\mu_0)}{1+\frac{22}{3}\frac{C_2(G)}{(4\pi)^2} g^2(\mu_0) \ln \frac{\mu}{\mu_0}} = \frac{1}{\frac{22}{3}\frac{C_2(G)}{(4\pi)^2} \ln \frac{\mu}{\Lambda_{\mathrm{QCD}}}}, \quad (5.6)$$

where we have used the boundary condition $g(\mu_0)=\infty$ at $\mu_0=\Lambda_{\mathrm{QCD}}$. The remaining two equations (5.5a) and (5.5b) cannot be solved exactly.

A. Fixed points

First, we obtain the fixed point of the RG. Note that the derivative $(1/g^2)\mu(\partial/\partial\mu)$ in Eqs. (5.5a), (5.5b) is rewritten as

$$\frac{1}{g^2}\mu \frac{\partial}{\partial \mu} = \frac{22}{3} \frac{C_2(G)}{(4\pi)^2} \ln \frac{\mu}{\Lambda_{\mathrm{QCD}}} \mu \frac{\partial}{\partial \mu} = \frac{22}{3} \frac{C_2(G)}{(4\pi)^2} \frac{\partial}{\partial \ln \ln \frac{\mu}{\Lambda_{\mathrm{QCD}}}}. \quad (5.7)$$

Then the fixed point (to one-loop order) is obtained by solving the algebraic equation simultaneously:

TABLE I. Eigenvalues and eigenvectors of the linearized RG equation where the lines II, III, IV are defined below. At the IR fixed point A and UV fixed point B, two eigenvalues are degenerate.

	Eigenvalue $\left(\frac{13}{3}\frac{g^2C_2(G)}{(4\pi)^2}\times\right)$		Eigenvector (ζ,λ)	(α,α')	
A	1	IR fixed point		(1, *a*)	any lines
B	−1	UV fixed point	(1, *a*)	(1, *a*)	any lines
C	1	Saddle point	(13, 3)	(1, −2)	line IV
	−1		(0, 1)	(0, 1)	line II
D	1	Saddle point	(−13, 3)	(0, 1)	line V
	−1		(1, −2)	(0, 1)	line III

$$\lambda\,\xi(\xi-1)\left(\xi-\frac{1}{2}\right)=0,\quad \lambda\left[\frac{13}{6}-\frac{\lambda}{2}+\lambda\,\xi(1-\xi)\right]=0. \tag{5.8}$$

We find one line of fixed points and three isolated fixed points in the (ξ,λ) plane or equivalently four isolated fixed points in the (α,α') plane.

(A) The line of fixed points $\lambda=0$, $\xi\in\mathbf{R}$ corresponds to an isolated fixed point $(\alpha,\alpha')=(0,0)$.

(B) $(\xi,\lambda)=(1/2,26/3)$ corresponds to $(\alpha,\alpha')=(26/3,0)$.

(C) $(\xi,\lambda)=(0,13/3)$ corresponds to $(\alpha,\alpha')=(0,13/3)$.

(D) $(\xi,\lambda)=(1,13/3)$ corresponds to $(\alpha,\alpha')=(26/3,-13/3)$.

If the two parameters ξ, λ are set equal to one of the fixed points, the theory remains forever on the fixed. If the system starts from other points and the scale μ is decreased, it evolves into the infrared (IR) region according to a couple of differential equations (5.5a)–(5.5c).

B. RG flow in the neighborhood of fixed points

In the neighborhood of the respective fixed point (X_1^*,X_2^*) in the plane $(X_1,X_2)=(\xi,\lambda)$ or (α,α'), we can study the behavior of the RG flow analytically. By taking into account only the terms which are linear in the infinitesimal deviation $\delta X_1:=X_1-X_1^*$, $\delta X_2:=X_2-X_2^*$ from the fixed point, a set of RG equations (5.5a) and (5.5b) is reduced to the form $\binom{\gamma_{X_1}}{\gamma_{X_2}}\sim\mathbf{A}\binom{\delta X_1}{\delta X_2}$, where $\mathbf{A}$ is a two by two matrix.

In the (ξ,λ) plane, the set of linearized RG equations reads

$$B\left(\frac{1}{2},\frac{26}{3}\right):\begin{pmatrix}\gamma_\xi\\ \gamma_\lambda\end{pmatrix}\sim-\frac{13}{3}\frac{g^2C_2(G)}{(4\pi)^2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\delta\xi\\ \delta\lambda\end{pmatrix}, \tag{5.9a}$$

$$C\left(0,\frac{13}{3}\right):\begin{pmatrix}\gamma_\xi\\ \gamma_\lambda\end{pmatrix}\sim\frac{13}{3}\frac{g^2C_2(G)}{(4\pi)^2}\begin{pmatrix}1&0\\ \frac{26}{3}&-1\end{pmatrix}\begin{pmatrix}\delta\xi\\ \delta\lambda\end{pmatrix}, \tag{5.9b}$$

$$D\left(1,\frac{13}{3}\right):\begin{pmatrix}\gamma_\xi\\ \gamma_\lambda\end{pmatrix}\sim\frac{13}{3}\frac{g^2C_2(G)}{(4\pi)^2}\begin{pmatrix}1&0\\ -\frac{26}{3}&-1\end{pmatrix}\begin{pmatrix}\delta\xi\\ \delta\lambda\end{pmatrix}. \tag{5.9c}$$

Similarly, in the (α,α') plane, we obtain

$$A(0,0):\begin{pmatrix}\gamma_\alpha\\ \gamma_{\alpha'}\end{pmatrix}\sim\frac{13}{3}\frac{g^2C_2(G)}{(4\pi)^2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\delta\alpha\\ \delta\alpha'\end{pmatrix}, \tag{5.10a}$$

$$B\left(\frac{26}{3},0\right):\begin{pmatrix}\xi_\alpha\\ \gamma_{\alpha'}\end{pmatrix}\sim-\frac{13}{3}\frac{g^2C_2(G)}{(4\pi)^2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\delta\alpha\\ \delta\alpha'\end{pmatrix}, \tag{5.10b}$$

$$C\left(0,\frac{13}{3}\right):\begin{pmatrix}\gamma_\alpha\\ \gamma_{\alpha'}\end{pmatrix}\sim\frac{13}{3}\frac{g^2C_2(G)}{(4\pi)^2}\begin{pmatrix}1&0\\-1&-1\end{pmatrix}\begin{pmatrix}\delta\alpha\\ \delta\alpha'\end{pmatrix}, \tag{5.10c}$$

$$D\left(-\frac{13}{3},\frac{26}{3}\right):\begin{pmatrix}\gamma_\alpha\\ \gamma_{\alpha'}\end{pmatrix}\sim\frac{13}{3}\frac{g^2C_2(G)}{(4\pi)^2}\begin{pmatrix}-1&0\\1&1\end{pmatrix}\begin{pmatrix}\delta\alpha\\ \delta\alpha'\end{pmatrix}. \tag{5.10d}$$

The respective matrix characterizing the behavior of the RG flow in the neighborhood of the respective fixed point has the eigenvalue and the corresponding eigenvector enumerated in Table I. The direction of the flow is determined at the respec-

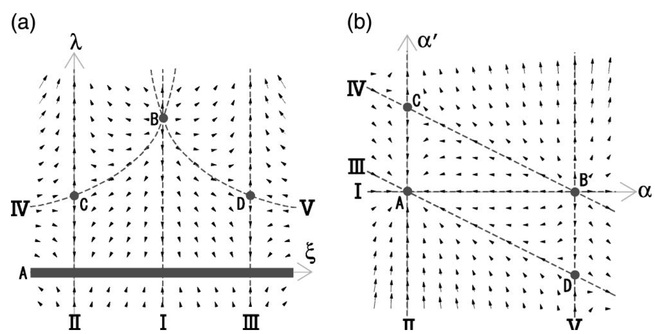


FIG. 4. RG flows in the (ξ,λ) plane (a) and in the (α,α') plane (b).

tive fixed point. We will see that these results are consistent with the global flow diagram given in Fig. 4 below.

C. Global behavior of the RG flow

We find that $\xi\equiv 0$, $\xi\equiv 1/2$, and $\xi\equiv 1$ are solutions of Eq. (5.5a). This implies that the RG flow starting from the point on one of the three planes $(0,\lambda,g)$, $(1/2,\lambda,g)$, $(1,\lambda,g)$, is always kept on the respective plane. On the three planes, moreover, Eq. (5.5b) can be solved exactly. On the plane $(1/2,\lambda,g)$, the RG flow in the region $0<\lambda<26/3$ obeys

$$\lambda(\mu)=\frac{26}{3}\left\{1+C\left(\ln\frac{\mu}{\Lambda_{\rm QCD}}\right)^{-13/22}\right\}^{-1}, \qquad (5.11)$$

where C is a positive constant. We see that λ approaches to the ultraviolet (UV) fixed point $\lambda\uparrow 26/3$ in the UV limit $\mu\uparrow\infty$, while $\lambda\downarrow 0$ monotonically as $\mu\downarrow\Lambda_{\rm QCD}$. On the other hand, the RG flow in the region $\lambda>26/3$ is described by

$$\lambda(\mu)=\frac{26}{3}\left\{1-C\left(\ln\frac{\mu}{\Lambda_{\rm QCD}}\right)^{-13/22}\right\}^{-1}, \qquad (5.12)$$

where λ approaches to the UV fixed point $\lambda\uparrow 26/3$ in the UV limit $\mu\uparrow\infty$, while $\lambda\uparrow\infty$ monotonically as $\mu\downarrow\Lambda_{\rm QCD}$. By substituting $\ln(\mu/\Lambda_{\rm QCD})=\{22/3C_2(G)/(4\pi)^2g^2\}^{-1}$ into the above equation, the equation of the RG flow on the plane $(1/2,\lambda,g)$ is obtained:

$$\lambda=\frac{26}{3}\left\{1\pm C\left(\frac{22}{3}\frac{C_2(G)}{(4\pi)^2}g^2\right)^{13/22}\right\}^{-1}. \qquad (5.13)$$

The RG flows on the plane $(0,\lambda,g)$ and $(1,\lambda,g)$ are governed by the same equations which are obtained by replacing 26/3 with 13/3.

The global behavior of the RG flow is obtained by solving Eqs. (5.5c)–(5.5b) numerically. In Fig. 4, the RG flow is drawn on the plane (ξ, λ) and the plane (α,α'). The direction of the arrow denotes the direction towards the IR region and the length of the arrow is proportional to the magnitude of the vector $\mu(d/d\mu)(\xi,\lambda)/g^2$. In the neighborhood of the respective fixed point, we see that the numerical result agrees with the analytical result given in Table I of the previous subsection.

Among the RG flows, the five RG flows (I, II, III, IV, V) connecting the fixed points A, B, C, D form the watershed (or backbone) in the flow diagram:

$$\text{(I)}\quad \xi=\frac{1}{2},\quad \alpha'=0, \qquad (5.14a)$$

$$\text{(II)}\quad \xi=0,\quad \alpha=0, \qquad (5.14b)$$

$$\text{(III)}\quad \xi=1,\quad \alpha'=-\frac{1}{2}\alpha, \qquad (5.14c)$$

$$\text{(IV)}\quad \lambda=\frac{13}{3}\frac{1}{1-\xi},\quad \alpha'=-\frac{1}{2}\alpha+\frac{13}{3}, \qquad (5.14d)$$

$$\text{(V)}\quad \lambda=\frac{13}{3}\frac{1}{\xi},\quad \alpha=\frac{26}{3}. \qquad (5.14e)$$

Since the flow is symmetric for the reflection with respect to the straight line I, $\xi=1/2$, we focus on the region $\xi\leqslant 1/2$. The flow starting from the initial position below IV runs towards the line A of fixed points and eventually arrive at A. If it arrives at a fixed point on A with a certain value of ξ depending on the initial position, then it does not move anymore. On the other hand, the flow starting from the initial position above IV runs away into the infinity, $\lambda=+\infty$. Here the flow on the line I and II is not an exception. However, it should be remarked that the fixed point B is IR repulsive in both directions, while the fixed point C is IR attractive on IV and repulsive on II. In view of these, it turns out that any fixed point on A is IR stable, while the fixed point B on I is a rather special fixed point which is IR unstable (UV stable).[5]

We have shown that the three fixed points B, C, D for the gauge parameter ξ, λ are located on lines I, II, III ($\xi=1/2,0,1$), respectively. On lines I, II, III, the RG flow is confined in the respective line; the Lagrangian takes the following form.

(I) $\xi=1/2$ (i.e., $\alpha\in\mathbf{R}$, $\alpha'=0$). The GF+FP term is invariant under the FP ghost conjugation and the orthosymplectic transformation OSp(4|2) [13]:

$$\mathcal{L}_{\rm GF+FP}=i\,\delta_B\bar{\delta}_B\left(\frac{1}{2}\mathcal{A}_\mu\cdot\mathcal{A}^\mu-\frac{\alpha}{2}i\mathcal{C}\cdot\bar{\mathcal{C}}\right). \qquad (5.15)$$

There is a four-ghost interaction.

(II) $\xi=0$ (i.e., $\alpha=0$, $\alpha'\in\mathbf{R}$). The GF+FP term is invariant under the global shift of antighost $\bar{\mathcal{C}}$:

$$\mathcal{L}_{\rm GF+FP}=\frac{\alpha'}{2}\mathcal{B}\cdot\mathcal{B}+\mathcal{B}\cdot\partial_\mu\mathcal{A}^\mu+i\bar{\mathcal{C}}\cdot\partial_\mu\mathcal{D}^\mu[\mathcal{A}]\mathcal{C}. \qquad (5.16)$$

There is no four-ghost interaction. This Lagrangian is the same as that in the conventional Lorentz gauge.

(III) $\xi=1$ (i.e., $\alpha'=-1/2\alpha$). The GF+FP term is invariant under the global shift of ghost $\mathcal{C}$:

$$\mathcal{L}_{\rm GF+FP}=\frac{\lambda}{2}\mathcal{B}\cdot\mathcal{B}+\mathcal{B}\cdot\partial_\mu\mathcal{A}^\mu+i\bar{\mathcal{C}}\cdot\mathcal{D}^\mu[\mathcal{A}]\partial_\mu\mathcal{C}. \qquad (5.17)$$

There is no four-ghost interaction. The choice II or III eliminates the four ghost interaction and the Yang-Mills theory reduces to the FP case. Once $\xi=0$ or $\xi=1$ is chosen, ξ is not renormalized by quantum corrections, since $\xi=0$ and $\xi=1$ are fixed point of the renormalization group. Then the FP Lagrangian is preserved under the renormalization.

In II and III, the role of ghost and antighost is interchanged. The FP ghost conjugation invariance is broken in the usual FP Lagrangian where the ghost and antighost are not treated on equal footing (except for the Landau gauge).

[5] This does not imply that a similar result is also obtained for the MA gauge. For example, $\alpha=0$ is not a fixed point in the MA gauge. See Ref. [27] for details.

In other words, the FP ghost conjugation invariance is recovered for $\alpha'=0$ (i.e., $\xi=1/2$ or $\lambda=\alpha$) by including the quartic ghost interaction even for $\alpha=0$.

We must keep in mind that these results are obtained to one-loop order. Therefore, the details of the flow diagram may change if we include higher-order corrections. The higher-order result is not known to date and will be given elsewhere. Nevertheless, the existence of the fixed point at $\lambda=0$ remains true to any finite order of perturbation. The existence of the lines I, II, and III are also guaranteed even after the inclusion of higher order terms, since it is protected by the symmetry dictated in the above. This is because the symmetry cannot be broken as far as the perturbation series to all orders are not summed up.

VI. RENORMALIZING THE COMPOSITE OPERATOR OF MASS DIMENSION 2

In this section we discuss the renormalization of the composite operator of mass dimension 2 and its BRST and anti-BRST invariance under the renormalization.

A. On-shell BRST transformation

By eliminating the Nakanishi-Lautrup field $\mathcal{B}$, the on-shell BRST and anti-BRST transformations are obtained as

$$\delta_B\bar{\mathcal{C}}(x)=i\left[-\frac{1}{\lambda}\partial^\mu\mathcal{A}_\mu(x)+\xi ig\mathcal{C}(x)\times\bar{\mathcal{C}}(x)\right], \tag{6.1}$$

$$\bar{\delta}_B\mathcal{C}(x)=i\left[\frac{1}{\lambda}\partial^\mu\mathcal{A}_\mu(x)-(\xi-1)ig\mathcal{C}(x)\times\bar{\mathcal{C}}(x)\right]. \tag{6.2}$$

The nilpotency of the on-shell transformation is partially broken[6] by the equation of motion of ghost and antighost:

$$(\delta_B)^2\mathcal{A}_\mu(x)=0, \tag{6.3a}$$

$$(\delta_B)^2\mathcal{C}(x)=0, \tag{6.3b}$$

$$(\delta_B)^2\bar{\mathcal{C}}(x)=\frac{-1}{\lambda}\frac{\delta\mathcal{L}_{\rm YM}^{\rm tot}}{\delta\bar{\mathcal{C}}}$$
$$=\frac{-1}{\lambda}[\partial^\mu\mathcal{D}_\mu\mathcal{C}-g\xi(\partial^\mu\mathcal{A}_\mu\times\mathcal{C}) +ig^2\lambda\xi(\xi-1)(\mathcal{C}\times\bar{\mathcal{C}})\times\mathcal{C}] \tag{6.3c}$$

and

$$(\bar{\delta}_B)^2\mathcal{A}_\mu(x)=0, \tag{6.4a}$$

$$(\bar{\delta}_B)^2\mathcal{C}(x)=\frac{-1}{\lambda}\frac{\delta\mathcal{L}_{\rm YM}^{\rm tot}}{\delta\mathcal{C}}$$
$$=\frac{i}{\lambda}[\partial^\mu\mathcal{D}_\mu\bar{\mathcal{C}}-g(1-\xi)(\partial^\mu\mathcal{A}_\mu\times\bar{\mathcal{C}}) -ig^2\lambda\xi(\xi-1)(\mathcal{C}\times\bar{\mathcal{C}})\times\bar{\mathcal{C}},] \tag{6.4b}$$

$$(\bar{\delta}_B)^2\bar{\mathcal{C}}(x)=0. \tag{6.4c}$$

Moreover, the anticommutatibity is also broken in a similar way:

$$(\delta_B\bar{\delta}_B+\bar{\delta}_B\delta_B)\mathcal{A}_\mu(x)=0, \tag{6.5a}$$

$$(\delta_B\bar{\delta}_B+\bar{\delta}_B\delta_B)\mathcal{C}(x)=\frac{1}{\lambda}\frac{\delta\mathcal{L}_{\rm YM}^{\rm tot}}{\delta\bar{\mathcal{C}}}, \tag{6.5b}$$

$$(\delta_B\bar{\delta}_B+\bar{\delta}_B\delta_B)\bar{\mathcal{C}}(x)=\frac{1}{\lambda}\frac{\delta\mathcal{L}_{\rm YM}^{\rm tot}}{\delta\mathcal{C}}. \tag{6.5c}$$

B. Composite operator of mass dimension 2

We define the composite operator $\mathcal{O}$ as a linear combination of two composite operators of mass dimension 2:

$$\mathcal{O}=(\Omega^{(D)})^{-1}\int d^Dx\left[\frac{1}{2}\mathcal{A}_\mu(x)\cdot\mathcal{A}^\mu(x)+\lambda i\bar{\mathcal{C}}(x)\cdot\mathcal{C}(x)\right]. \tag{6.6}$$

The on-shell BRST transformation of the operator $\mathcal{O}$ is calculated as

$$\delta_B\mathcal{O}=(\Omega^{(D)})^{-1}\int d^Dx\,\delta_B\left[\frac{1}{2}\mathcal{A}_\mu(x)\cdot\mathcal{A}^\mu(x)+\lambda i\bar{\mathcal{C}}(x)\cdot\mathcal{C}(x)\right]$$
$$=(\Omega^{(D)})^{-1}\int d^Dx[\mathcal{A}_\mu(x)\cdot\delta_B\mathcal{A}^\mu(x)-\lambda i\bar{\mathcal{C}}(x)\cdot\delta_B\mathcal{C}(x) +\lambda i\,\delta_B\bar{\mathcal{C}}(x)\cdot\mathcal{C}(x)]$$

[6]An elegant proof of the unitarity of gauge theory is given based on the nilpotency of the BRST transformation, see, e.g., Ref. [30]. The nilpotency is indeed broken in the on-shell BRST transformation which is obtained by eliminating the NL field. However, the nilpotency is not the only way to show the unitarity. Even in this case, it is possible to show the unitarity order by order of perturbation theory based on the Feynman diagrams without the NL fields.

$$=(\Omega^{(D)})^{-1}\int d^Dx\bigg[\mathcal{A}_\mu(x)\cdot\partial^\mu\mathcal{C}(x)$$
$$+\lambda i\bar{\mathcal{C}}(x)\cdot\frac{g}{2}[\mathcal{C}(x)\times\mathcal{C}(x)]+\partial^\mu\mathcal{A}_\mu(x)\cdot\mathcal{C}(x)$$
$$-\lambda\xi ig[\mathcal{C}(x)\times\bar{\mathcal{C}}(x)]\cdot\mathcal{C}(x)\bigg]$$
$$=(\Omega^{(D)})^{-1}\int d^Dx\bigg\{\partial^\mu[\mathcal{A}_\mu(x)\cdot\mathcal{C}(x)]+\lambda\left(\frac{1}{2}-\xi\right)$$
$$\times i\bar{\mathcal{C}}(x)\cdot g[\mathcal{C}(x)\times\mathcal{C}(x)]\bigg\}. \tag{6.7}$$

In a similar way, the on-shell anti-BRST transformation of the operator $\mathcal{O}$ is calculated as

$$\bar{\delta}_{\rm B}\mathcal{O}=(\Omega^{(D)})^{-1}\int d^Dx\bigg\{\partial^\mu[\mathcal{A}_\mu(x)\cdot\bar{\mathcal{C}}(x)]$$
$$+\lambda\left(\frac{1}{2}-\xi\right)i\bar{\mathcal{C}}(x)\cdot g[\bar{\mathcal{C}}(x)\times\mathcal{C}(x)]\bigg\}. \tag{6.8}$$

Therefore, the composite operator $\mathcal{O}$ is invariant under the BRST and anti-BRST transformations when

$$\xi=\frac{1}{2}\ \text{ or }\ \lambda=0, \tag{6.9}$$

i.e., on the line I and A in the (ξ,λ) plane, or on the line I in the (α,α') plane. For $\xi=1/2$, the on-shell BRST and anti-BRST transformations read

$$\delta_{\rm B}\bar{\mathcal{C}}(x)=-\frac{i}{\alpha}\partial^\mu\mathcal{A}_\mu(x)-\frac{1}{2}g\mathcal{C}(x)\times\bar{\mathcal{C}}(x), \tag{6.10}$$

$$\bar{\delta}_{\rm B}\mathcal{C}(x)=+\frac{i}{\alpha}\partial^\mu\mathcal{A}_\mu(x)-\frac{1}{2}g\mathcal{C}(x)\times\bar{\mathcal{C}}(x). \tag{6.11}$$

The special case $\lambda=0$ (and $\alpha=0$ to have a finite ξ) is nothing but the Landau gauge in the conventional Lorentz gauge and the BRST and anti-BRST invariant operator $\mathcal{O}$ reduces to the simple form

$$\mathcal{O}'=(\Omega^{(D)})^{-1}\int d^Dx\left[\frac{1}{2}\mathcal{A}_\mu(x)\cdot\mathcal{A}^\mu(x)\right]. \tag{6.12}$$

Note that $\mathcal{O}'$ is invariant under the gauge transformation as well as the BRST and anti-BRST transformations.

C. Renormalization of the composite operator

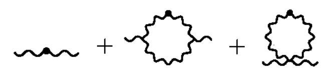
Hereafter, we use the following notation to simplify the expressions:

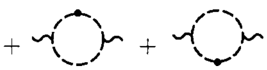
$$A:=\mathcal{A}^R,\ \ C:=\mathcal{C}^R,\ \ \bar{C}:=\bar{\mathcal{C}}^R,\ \ B:=\mathcal{B}^R. \tag{6.13}$$

We consider the Green function of the fundamental fields with the insertion of a composite operator of mass dimension 2. In the following, it is assumed that we have already finished the renormalization for the fundamental field in the perturbative theory. Therefore, we only have to consider the extra renormalization for the divergence coming from the inserted composite operators in the renormalized Green function. In order to take into account operator mixing among composite operators with the same mass dimension and the same quantum number, we must introduce the matrix of renormalization factors $Z_1,\ldots,Z_4$:

$$\begin{pmatrix}\left[\frac{1}{2}AA\right]_R\\ [i\bar{C}C]_R\end{pmatrix}=\begin{pmatrix}Z_1 & Z_2\\ Z_3 & Z_4\end{pmatrix}\begin{pmatrix}\left[\frac{1}{2}AA\right]\\ [i\bar{C}C]\end{pmatrix}. \tag{6.14}$$

Then, to the lowest nontrivial order, we find

$\left\langle AA\left[\frac{1}{2}AA\right]\right\rangle=$  $+\cdots,$ (6.15a)

$\left\langle AA\left[i\bar{C}C\right]\right\rangle=\ 0\ +$  $+\cdots,$ (6.15b)

$\left\langle i\bar{C}C\left[\frac{1}{2}AA\right]\right\rangle=\ 0\ +$  $+\cdots,$ (6.15c)

$\left\langle i\bar{C}C\left[i\bar{C}C\right]\right\rangle=$  $+\cdots,$ (6.15d)

where we have used the Feynman rule

 $=\delta^{AB},$ (6.16a)

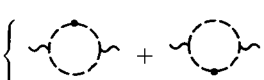
 $=i\delta^{AB},$ (6.16b)

with the dot denoting the insertion of a composite operator.

We show that the divergences coming from the compositeness are absorbed by taking the four renormalization constants Z_1, Z_2, Z_3, Z_4 appropriately. The first example is

$\left\langle AA\left[\frac{1}{2}AA\right]_{\rm R}\right\rangle=Z_1\left\langle AA\left[\frac{1}{2}AA\right]\right\rangle+Z_2\left\langle AA\left[i\bar{C}C\right]\right\rangle$

$=Z_1\{$  $+\cdots\}$

$+Z_2\{$  $+\cdots\}$

$\equiv$  .

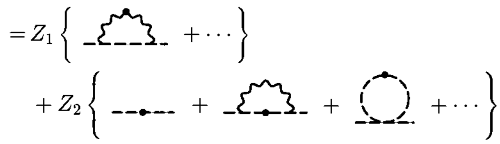
(6.17)

Hence the lowest value of Z_1 is 1:

$$Z_1 = 1 + Z_1^{(1)} + \cdots . \tag{6.18}$$

The second example is

$$\left\langle i\bar{C}C\left[\tfrac{1}{2}AA\right]_{\mathrm{R}}\right\rangle = Z_1\left\langle i\bar{C}C\left[\tfrac{1}{2}AA\right]\right\rangle + Z_2\left\langle i\bar{C}C[i\bar{C}C]\right\rangle$$



$\equiv 0$ (no divergence).

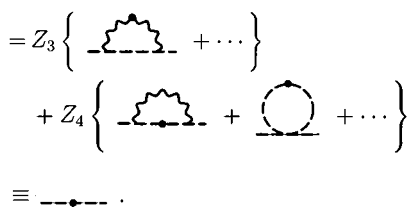
(6.19)

Hence Z_2 does not have the tree part and begins with the one-loop order

$$Z_2 = Z_2^{(1)} + \cdots . \tag{6.20}$$

The third example is

$$\left\langle i\bar{C}C\left[i\bar{C}C\right]_{\mathrm{R}}\right\rangle = Z_3\left\langle i\bar{C}C\left[\tfrac{1}{2}AA\right]\right\rangle + Z_4\left\langle i\bar{C}C\left[i\bar{C}C\right]\right\rangle$$



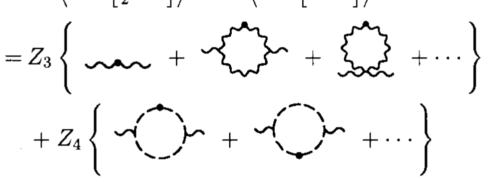
(6.21)

Hence, Z_4 has the form

$$Z_4 = 1 + Z_4^{(1)} + \cdots . \tag{6.22}$$

The fourth example is

$$\left\langle AA\left[i\bar{C}C\right]_{\mathrm{R}}\right\rangle = Z_3\left\langle AA\left[\tfrac{1}{2}AA\right]\right\rangle + Z_4\left\langle AA\left[i\bar{C}C\right]\right\rangle$$



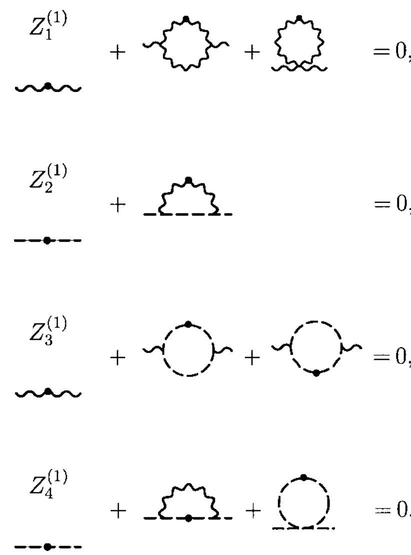
$\equiv 0$ (no divergence).

(6.23)

Hence, Z_3 begins with the one-loop order

$$Z_3 = Z_3^{(1)} + \cdots . \tag{6.24}$$

Therefore, up to one-loop order, the renormalization constants must satisfy the relationship



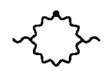
(6.25a)

(6.25b)

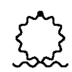
(6.25c)

(6.25d)

The explicit calculations lead to the following divergent parts:



$$\sim C_2(G)\delta^{AB}\left[3 + \frac{3}{4}\lambda(1+\lambda)\right] g_{\mu\nu}\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{6.26}$$



$$\sim -3C_2(G)\delta^{AB}g_{\mu\nu}\frac{3+\lambda^2}{4}\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{6.27}$$

$$\sim iC_2(G)\delta^{AB}\xi(1-\xi)\lambda^2\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{6.28}$$

$$\sim -\frac{1}{4}C_2(G)\delta^{AB}g_{\mu\nu}\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{6.29}$$

$$\sim iC_2(G)\delta^{AB}\xi(1-\xi)\lambda\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{6.30}$$

$$\sim -iC_2(G)\delta^{AB}\xi(1-\xi)\lambda\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}. \tag{6.31}$$

Thus the renormalization constants for the composite operators are obtained as

$$Z_1^{(1)} = -\frac{3}{4}(1+\lambda)C_2(G)\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{6.32a}$$

$$Z_2^{(1)} = -\lambda^2\xi(1-\xi)C_2(G)\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{6.32b}$$

$$Z_3^{(1)} = \frac{1}{2}C_2(G)\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{6.32c}$$

$$Z_4^{(1)} = 0. \tag{6.32d}$$

We pay attention to the renormalization constants of composite operators in light of the inverted relation of Eq. (6.14):

$$\begin{pmatrix} \left[\frac{1}{2}AA\right] \\ [i\bar{C}C] \end{pmatrix} = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}^{-1} \begin{pmatrix} \left[\frac{1}{2}AA\right]_R \\ [i\bar{C}C]_R \end{pmatrix} = \begin{pmatrix} 1-Z_1^{(1)} & -Z_2^{(1)} \\ -Z_3^{(1)} & 1-Z_4^{(1)} \end{pmatrix} \begin{pmatrix} \left[\frac{1}{2}AA\right]_R \\ [i\bar{C}C]_R \end{pmatrix}. \tag{6.33}$$

This relation shows that there is an operator mixing between the gluon and ghost composite operators which are of mass dimension 2 and color singlet, as pointed out in Ref. [2]. In the absence of four-ghost interaction ($\xi=0$ or $\xi=1$), Eqs. (6.28), (6.30), and (6.31) vanish and hence we have $Z_2^{(1)}=0=Z_4^{(1)}$. In this case, there is no contribution from ghost for the renormalization of the gluon composite operator $[1/2AA]$

$$\left[\frac{1}{2}AA\right] = (1-Z_1^{(1)})\left[\frac{1}{2}AA\right]_R, \tag{6.34}$$

$$[i\bar{C}C] = [i\bar{C}C]_R - Z_3^{(1)}\left[\frac{1}{2}AA\right]_R. \tag{6.35}$$

On the other hand, the ghost composite operator cannot be finite without the mixing of the gluon composite operator. In the conventional Lorentz gauge fixing, therefore, we do not have to consider the contribution from ghost in treating the renormalization of the gluon composite operator $[1/2AA]$ (at least in the one-loop level).

D. Multiplicative renormalizability of the composite operator

Now we examine the multiplicative renormalizability of the composite operator $\mathcal{O}$. Taking into account the renormalization of the fundamental field and the composite field (6.33), we obtain

$$\begin{aligned} Q_0 &:= \frac{1}{2}A_0A_0 + \lambda_0 i\bar{C}_0C_0 \\ &= (1+Z_A^{(1)})\frac{1}{2}AA + (1+Z_\lambda^{(1)})(1+Z_C^{(1)})\lambda i\bar{C}C \\ &= (1+Z_A^{(1)})\left\{(1-Z_1^{(1)})\left[\frac{1}{2}AA\right]_R - Z_2^{(1)}[i\bar{C}C]_R\right\} \\ &\quad + (1+Z_\lambda^{(1)})(1+Z_C^{(1)})\lambda\left\{-Z_3^{(1)}\left[\frac{1}{2}AA\right]_R + (1-Z_4^{(1)})[i\bar{C}C]_R\right\} \\ &= \{1+Z_A^{(1)}-Z_1^{(1)}-\lambda Z_3^{(1)}\}\left[\frac{1}{2}AA\right]_R \\ &\quad + \left\{-\frac{Z_2^{(1)}}{\lambda}+1+Z_\lambda^{(1)}+Z_C^{(1)}-Z_4^{(1)}\right\}\lambda[i\bar{C}C]_R. \end{aligned} \tag{6.36}$$

The multiplicative renormalizability holds (in the one-loop level) if and only if

$$Z_Q^{(1)} := Z_A^{(1)} - Z_1^{(1)} - \lambda Z_3^{(1)} = -\frac{Z_2^{(1)}}{\lambda} + Z_\lambda^{(1)} + Z_C^{(1)} - Z_4^{(1)}. \tag{6.37}$$

This is equivalent to the condition

$$\lambda\left(\xi - \frac{1}{2}\right)^2 = 0. \tag{6.38}$$

If this condition is satisfied, the composite operator is multiplicatively renormalized as

$$Q_0 = Z_Q\left(\left[\frac{1}{2}AA\right]_R + \lambda[i\bar{C}C]_R\right), \tag{6.39}$$

$$Z_Q^{(1)} = \left(\frac{35}{12} - \frac{1}{4}\lambda\right)C_2(G)\frac{(g\mu^{-\epsilon})^2}{(4\pi)^2}\frac{1}{\epsilon}. \tag{6.40}$$

In the case of $\lambda=0$, this result reduces to that of Boucaud *et al.* [22] without operator mixing.

It should be remarked that the composite operator is not multiplicatively renormalizable, unless the renormalization of the composite operators AA and $\bar{C}C$ are taken into account. In fact, the multiplicative renormalizability of

$$\begin{aligned} Q_0 &:= \frac{1}{2}A_0A_0 + \lambda_i\bar{C}_0C_0 \\ &= (1+Z_A^{(1)})\frac{1}{2}AA + (1+Z_\lambda^{(1)}+Z_C^{(1)})\lambda i\bar{C}C + O(\hbar^2), \end{aligned} \tag{6.41}$$

without the renormalization of the composite operator leads to the condition $Z_A^{(1)}-Z_\lambda^{(1)}-Z_C^{(1)}=0$, which reads $\lambda[\xi(\xi-1)+1/4]=3/4$. This curve does not have a definite meaning in the renormalization, since the curve is not along the RG flow.

E. BRST invariance of the renormalized composite operator

Finally, we show that the renormalized composite operator $\mathcal{O}^R$ is invariant under the renormalized BRST and anti-BRST transformations. By requiring that the renormalized BRST and anti-BRST transformations are nilpotent and anticommute:

$$\delta_B^R\delta_B^R \equiv 0, \quad \bar{\delta}_B^R\bar{\delta}_B^R \equiv 0, \quad \delta_B^R\bar{\delta}_B^R + \bar{\delta}_B^R\delta_B^R \equiv 0, \tag{6.42}$$

the renormalized BRST and anti-BRST transformations for the renormalized fields A_μ, C, $\bar{C}$, B are determined (by an appropriate rescaling of $\mathcal{B}$ field) as [8,9]

$$\delta_B^R A_\mu(x) = X\mathcal{D}_\mu[A]^R C(x) := X\{\partial_\mu C(x) + Z_A^{1/2} Z_g g_R [A_\mu(x) C(x)]\}, \tag{6.43a}$$

$$\delta_B^R C(x) = -\frac{1}{2} X Z_A^{1/2} Z_g g_R [C(x)\times C(x)] \tag{6.43b}$$

$$\delta_B^R \bar{C}(x) = iXB(x), \tag{6.43c}$$

$$\delta_B^R B(x) = 0, \tag{6.43d}$$

and

$$\bar{\delta}_B^R A_\mu(x) = \bar{X}\mathcal{D}_\mu[A]^R \bar{C}(x) := \bar{X}\{\partial_\mu \bar{C}(x) + Z_A^{1/2} Z_g g_R [A_\mu(x)\times \bar{C}(x)]\}, \tag{6.44a}$$

$$\bar{\delta}_B^R \bar{C}(x) = -\frac{1}{2}\bar{X} Z_A^{1/2} Z_g g_R [\bar{C}(x)\times \bar{C}(x)] \tag{6.44b}$$

$$\bar{\delta}_B^R C(x) = i\overline{XB}(x), \tag{6.44c}$$

$$\bar{\delta}_B^R \bar{B}(x) = 0, \tag{6.44d}$$

where X and $\bar{X}$ are arbitrary real numbers and $\bar{B}$ is defined by

$$\bar{B}(x) = -B(x) + iZ_A^{1/2} Z_g g_R [C(x)\times \bar{C}(x)]. \tag{6.45}$$

The Lagrangian is written by making use of the renormalized BRST and anti-BRST transformations and the renormalized fields as

$$\mathcal{L}_{\rm YM}^{\rm tot} = -\frac{1}{4} Z_A(\partial_\mu A_\nu - \partial_\nu A_\mu + Z_g Z_A^{1/2} g_R A_\mu \times A_\nu)^2 + \frac{Z_C}{X\bar{X}} i\delta_B^R \bar{\delta}_B^R \left(\frac{1}{2}A_\mu\cdot A^\mu - \frac{Z_C Z_\alpha}{Z_A}\frac{\alpha_R}{2} iC\cdot\bar{C}\right) + \frac{Z_C^2 Z_{\alpha'}}{Z_A}\frac{\alpha'_R}{2} B\cdot B. \tag{6.46}$$

This agrees with Eq. (4.21).

We derive the condition for the renormalized composite operator $\mathcal{O}_R$ to be invariant under the renormalized BRST transformation defined above. We can write a finite composite operator of mass dimension 2 in the form (up to an overall constant):

$$\mathcal{Q}_R = \left[\frac{1}{2}A_\mu(x)\cdot A^\mu(x)\right]_R + K_R[i\bar{C}(x)\cdot C(x)]_R, \tag{6.47}$$

where K_R is a finite function of the renormalized parameters g_R, ξ_R, λ. Performing the renormalized BRST transformation (6.43d) after the renormalization factors (6.33) of the composite operator are included, we obtain

$$\begin{aligned}\delta_B^R \mathcal{Q}_R &= \delta_B^R\left\{(Z_1+K_R Z_3)\left(\frac{1}{2}A_\mu\cdot A^\mu\right) + (Z_2+K_R Z_4)(i\bar{C}\cdot C)\right\}\\ &= (Z_1+K_R Z_3) X\partial_\mu C\cdot A^\mu + (Z_2+K_R Z_4)\\ &\quad\times\left\{i\bar{C}\cdot\left(XZ_A^{1/2}Z_g\frac{g}{2}C\times C\right) + X\left(\frac{Z_A}{Z_C Z_\lambda}\frac{1}{\lambda}\partial_\mu A^\mu - iZ_A^{1/2}Z_\xi \xi Z_g g C\times\bar{C}\right)\cdot C\right\}.\end{aligned} \tag{6.48}$$

For the right-hand side to be a total derivative, we must require two conditions: (1) the coefficient for the term $C\cdot(\bar{C}\times C)$ vanishes, (2) the remaining terms containing the derivative are combined into a total derivative term. The respective condition reads

$$\frac{Z_A^{1/2}Z_g}{2} = Z_A^{1/2} Z_g Z_\xi \xi, \tag{6.49}$$

$$Z_1 + K_R Z_3 = (Z_2 + K_R Z_4)\frac{Z_A}{Z_C Z_\lambda}\frac{1}{\lambda}. \tag{6.50}$$

The first condition reduces to

$$\xi_0 = Z_\xi \xi = \frac{1}{2}. \tag{6.51}$$

Since Z_2, $Z_3 \sim O(\hbar/\epsilon)$ and Z_1, $Z_4 \sim 1 + O(\hbar/\epsilon)$, the second condition yields for the $O(1)$ term

$$K_R = \lambda_R, \tag{6.52}$$

and for the $O(1/\epsilon)$ term

$$Z_A^{(1)} - Z_1^{(1)} - \lambda_R Z_3^{(1)} + \frac{Z_2^{(1)}}{\lambda_R} - Z_\lambda^{(1)} - Z_C^{(1)} + Z_4^{(1)} = 0. \tag{6.53}$$

This condition is the same as Eq. (6.37). In the Landau gauge $\alpha=\lambda=0$, especially, the condition (6.53) reduces to $Z_2^{(1)} = 0$. This is automatically satisfied in this case.

VII. OPERATOR PRODUCT EXPANSION AND VACUUM CONDENSATE

We apply the operator product expansion or short distance expansion (SDE) to the gluon and ghost propagators. The OPE was originally proposed as an operator relation by Wilson [46]. For example, the product of two scalar field operators defined at different spacetime points is expanded as

$$\phi(x)\phi(y)\sim\sum_i F^{[\mathcal{O}_i]}(x-y)\left[\mathcal{O}_i\left(\frac{x+y}{2}\right)\right], \tag{7.1}$$

where the composite operators $\{\mathcal{O}_i\}$ form a complete set of renormalized local operators. The famous proof of OPE by Zimmermann [47] was given in the framework of perturbation theory. Quite recently, the OPE was rigorously proved as an operator relation by Bostelman [48].[7] According to the method [49,50], the (Fourier transformed) Wilson coefficient $\tilde{F}^{[\phi_1\cdots\phi_n]}(p)$ in the OPE

$$\phi(x)\phi(y)\sim\sum_n F^{[\phi_1\cdots\phi_n]}(x-y)\left[\phi_1\cdots\phi_n\left(\frac{x+y}{2}\right)\right] \tag{7.2}$$

can be calculated in perturbation theory by equating a $(2+n)$-point one-particle irreducible (1PI) Green's function—where two of the external legs have hard momentum p and the remaining n external legs are assigned zero momentum $q=0$—with the Wilson coefficient times an n point Green's function with an insertion of the relevant composite operator at zero momentum.

A. The OPE in the tree level

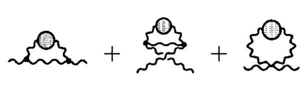
First, we consider the OPE of the inverse gluon propagator

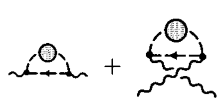
$$(D^{-1})^{AB}_{\mu\nu}(p)=C^{[1]AB}_{\mu\nu}(p)\langle 1\rangle+C^{[A^2]AB}_{\mu\nu}(p)\left\langle\frac{1}{2}A_\rho\cdot A^\rho\right\rangle + C^{[\bar{C}C]AB}_{\mu\nu}(p)\langle i\bar{C}\cdot C\rangle+\cdots, \tag{7.3}$$

where the first Wilson coefficient is nothing but the bare inverse gluon propagator

$$C^{[1]AB}_{\mu\nu}(p)=(D_0^{-1})^{AB}_{\mu\nu}(p):=-p^2(P^T_{\mu\nu}+\lambda^{-1}P^L_{\mu\nu})\delta^{AB} = -p^2\left(g_{\mu\nu}-\frac{p^\mu p^\nu}{p^2}+\lambda^{-1}\frac{p^\mu p^\nu}{p^2}\right)\delta^{AB}. \tag{7.4}$$

The other Wilson coefficients are calculated in the perturbation theory from the diagrams

$iC^{[A^2]}_{\mu\nu} =$  (7.5)

$C^{[\bar{C}C]}_{\mu\nu} =$  (7.6)

In these diagrams, two external legs have hard momentum p and the $(n=2)$ lines connected to a blob correspond to the external legs with zero momentum $q=0$.

The explicit calculation in the tree level yields the result (see the Appendix for the details of the calculations):

$$C^{[A^2]AB}_{\mu\nu}(p)=-\frac{N_c g^2}{2(N_c^2-1)}(1+\lambda)P^T_{\mu\nu}\delta^{AB}, \tag{7.7}$$

$$C^{[\bar{C}C]AB}_{\mu\nu}(p)=2\frac{N_c g^2}{(N_c^2-1)}\xi(1-\xi)P^L_{\mu\nu}\delta^{AB}, \tag{7.8}$$

where we have put $C_2(G)=N_c$ for simplicity. Defining the vacuum polarization tensor of the gluon by

$$(D^{-1})^{AB}_{\mu\nu}(p):=(D_0^{-1})^{AB}_{\mu\nu}(p)+\Pi^{AB}_{\mu\nu}(p), \tag{7.9}$$

we obtain the vacuum polarization tensor of the gluon

$$\Pi^{AB}_{\mu\nu}(p)=\frac{N_c g^2}{4(N_c^2-1)}\delta^{AB}\{-(1+\lambda)P^T_{\mu\nu}\langle A_\rho\cdot A^\rho\rangle + 2D\xi(1-\xi)P^L_{\mu\nu}\langle i\bar{C}\cdot C\rangle\}. \tag{7.10}$$

It turns out that even the inclusion of the quartic ghost interaction does not affect the Wilson coefficient $C^{[A^2]}_{\mu\nu}$, at least in the tree level. For the Wilson coefficient $C^{[\bar{C}C]}_{\mu\nu}$, however, there is an extra contribution coming from the quartic ghost interaction, as suggested already in Ref. [2]. The nonzero Wilson coefficient $C^{[\bar{C}C]}$ due to the presence of the quartic ghost interaction $(\xi\neq 0,1)$ breaks the transversality of the gluon polarization tensor, i.e., $\Pi_{\mu\nu}\neq P^T_{\mu\nu}\Pi$. This result does not contradict the Slavnov-Taylor identity [5,8,27]. When $\xi=0$ $(\xi=1)$, the ghost condensate $\langle i\bar{C}\cdot C\rangle$ cannot appear in the OPE, since the gluon-ghost-antighost vertex (4.16) is proportional to the outgoing ghost (antighost) momentum p_μ (q_μ). The above result (7.10) suggests the existence of the effective gluon mass given by

$$m_A^2=-\frac{N_c g^2}{4(N_c^2-1)}(1+\lambda)\langle A_\rho\cdot A^\rho\rangle. \tag{7.11}$$

Therefore, the gluon condensation of mass dimension 2 can be an origin of the gluon mass. The effect of higher orders will be investigated in the next subsection.

Next, we perform the OPE for the inverse ghost propagator

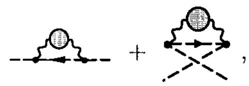
$$-i(G^{-1})^{AB}(p)=C^{[1]}_{AB}(p)\langle 1\rangle+C^{[A^2]}_{AB}(p)\left\langle\frac{1}{2}A_\rho\cdot A^\rho\right\rangle + C^{[\bar{C}C]}_{AB}(p)\langle i\bar{C}\cdot C\rangle+\cdots, \tag{7.12}$$

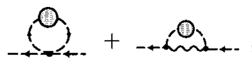
where the first Wilson coefficient agrees with the bare inverse ghost propagator

$$C^{[1]}_{AB}(p)=-i(G_0^{-1})^{AB}(p)=-p^2\delta^{AB}. \tag{7.13}$$

[7] The authors would like to thank Izumi Ojima for informing us of this reference.

The other Wilson coefficients are calculated from the diagrams

$$-C_{\rm gh}^{[A^2]} = $$  $$\qquad (7.14)$$

$$iC_g^{[\bar{C}C]} = $$  $$\qquad (7.15)$$

which yield the result

$$C_{AB}^{[A^2]}(p)=\frac{N_c g^2}{2(N_c^2-1)}\,\delta^{AB}, \qquad (7.16)$$

$$C_{AB}^{[\bar{C}C]}(p)=0. \qquad (7.17)$$

Here the coefficient $C_{AB}^{[\bar{C}C]}$ vanishes due to cancellation, see the Appendix. Defining the vacuum polarization tensor of the ghost by

$$(G^{-1})^{AB}(p)\coloneqq(G_0^{-1})^{AB}(p)+i\Pi_{gh}^{AB}(p), \qquad (7.18)$$

the vacuum polarization for the ghost is obtained:

$$\Pi_{gh}^{AB}(p)=\frac{N_c g^2}{4(N_c^2-1)}\,\delta_{AB}\langle A_\rho\cdot A^\rho\rangle. \qquad (7.19)$$

We find that the ghost vacuum polarization has no contribution from the ghost-antighost condensation even for $\xi\neq 0,1$. Thus we obtain the effective ghost mass

$$m_C^2=\frac{N_c g^2}{4(N_c^2-1)}\langle A_\rho\cdot A^\rho\rangle. \qquad (7.20)$$

This result shows that the gluon condensation of mass dimension 2 can also be an origin of the ghost mass.[8]

The combination of gluon and ghost condensation appearing in the OPE is not BRST invariant in the sense explained in the previous section. This is reasonable, since even the OPE of gauge invariant operators does not give a gauge invariant combination in the OPE, see, e.g., Ref. [51].

B. RG improvement of the OPE

One of the advantages of the OPE is that the momentum dependence of the Wilson coefficient is determined by the renormalization group equation. More accurately, the change of the Wilson coefficient under the RG transformation can be specified by the renormalization factors Z which are to be calculated before the RG improvement of the OPE calculus. Therefore, we can obtain higher-order corrections for the momentum dependence of the coefficient without any explicit higher-order computations (at least for the leading logarithmic corrections).

1. RG equation for Wilson coefficients

We begin with an OPE relation in the momentum representation obtained by extracting composite operators up to mass dimension 2 (we omit all indices, since they are not essential in the following arguments):

$$-i\tilde{A}_R(p)\tilde{A}_R(-p)=D_{\rm pert}(p)[1]+F_1^A(p)\left[\frac{1}{2}A(0)A(0)\right]_R+F_2^A(p)[i\bar{C}(0)C(0)]_R+\cdots, \qquad (7.21a)$$

$$\tilde{\bar{C}}_R(p)\tilde{C}_R(-p)=-iG_{\rm pert}(p)[1]+F_1^C(p)\left[\frac{1}{2}A(0)A(0)\right]_R+F_2^C(p)[i\bar{C}(0)C(0)]_R+\cdots, \qquad (7.21b)$$

where $D_{\rm pert}(p)$ and $G_{\rm pert}$ denote the perturbative gluon and ghost propagators, respectively, with the perturbative loop corrections included in addition to the OPE contribution.

First, we try to rewrite all field operators in both sides of Eqs. (7.21a) and (7.21b) in terms of bare quantities. Hereafter it is supposed that the Wilson coefficient and composite operators are defined based on the renormalization scheme depending on a certain parameter μ (corresponding to the mass scale), which is different from the Bogolubov-Paresiok-Hepp-Zimmermann (BPHZ) prescription at zero momentum $q=0$. In the actual calculations, we adopt the minimal subtraction (MS) scheme, although the resulting expressions can be translated into the momentum-space subtraction (MOM) scheme.

By making use of the Z factors calculated in the previous section, two OPE relations above are combined into a matrix form

$$\mathbf{Z}_f^{-1}\begin{pmatrix}-i\tilde{A}_0(p)\tilde{A}_0(-p)\\ \tilde{\bar{C}}_0(p)\tilde{C}_0(-p)\end{pmatrix}=\mathbf{D}_{\rm pert}+\mathbf{F}\tilde{\mathbf{Z}}\begin{pmatrix}\frac{1}{2}A_0(0)A_0(0)\\ i\bar{C}_0(0)C_0(0)\end{pmatrix}+\cdots, \qquad (7.22)$$

where we have defined the two by two matrices

$$\mathbf{Z}_f=\begin{pmatrix}Z_A & 0\\ 0 & Z_C\end{pmatrix},\quad \mathbf{F}\coloneqq\begin{pmatrix}F_1^A(p) & F_2^A(p)\\ F_1^C(p) & F_2^C(p)\end{pmatrix},$$

[8] In the Lorenz gauge, the effective gluon mass and ghost mass are generated by the gluon condensation of mass dimension 2 alone in the tree level. This is not the case if we include the high-order correction as will be shown in the next subsection. In the MA gauge, on the contrary, two condensations from the off-diagonal gluon and off-diagonal ghost contribute to the effective off-diagonal gluon and ghost masses already in the tree level, see Refs. [2,20].

$$\tilde{\mathbf{Z}}=\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}\mathbf{Z}_f^{-1}, \tag{7.23}$$

and a column vector

$$\mathbf{D}_{\mathrm{pert}}:=\begin{pmatrix} D_{\mathrm{pert}}(p) \\ -iG_{\mathrm{pert}}(p) \end{pmatrix}. \tag{7.24}$$

Introducing a matrix $\mathbf{F}_0$ by[9]

$$\mathbf{F}_0=\mathbf{Z}_f\mathbf{F}\tilde{\mathbf{Z}}:=\begin{pmatrix} F^A_{01}(p) & F^A_{02}(p) \\ F^C_{01}(p) & F^C_{02}(p) \end{pmatrix}, \tag{7.25}$$

we obtain an OPE relation among the bare quantities:

$$\begin{pmatrix} -i\tilde{A}_0(p)\tilde{A}_0(-p) \\ \tilde{\bar{C}}_0(p)\tilde{C}_0(-p) \end{pmatrix} =\mathbf{Z}_f\mathbf{D}_{\mathrm{pert}}+\mathbf{F}_0\begin{pmatrix} \frac{1}{2}A_0(0)A_0(0) \\ i\bar{C}_0(0)C_0(0) \end{pmatrix}+\cdots. \tag{7.26}$$

Second, we observe that the relation (7.26) should have no dependence on the renormalization point μ. Hence, the first term on the right-hand side of Eq. (7.26) is independent of μ, i.e.,

$$\mu\frac{d}{d\mu}(\mathbf{Z}_f\mathbf{D}_{\mathrm{pert}})=0, \tag{7.27}$$

and the coefficient $\mathbf{F}_0$ in the second term is also independent of μ, i.e.,

$$\mu\frac{d}{d\mu}\mathbf{F}_0=\mu\frac{d}{d\mu}(\mathbf{Z}_f\mathbf{F}\tilde{\mathbf{Z}})=0. \tag{7.28}$$

We multiply Eq. (7.28) by $\mathbf{Z}_f^{-1}$ from the left and by $\tilde{\mathbf{Z}}^{-1}$ from the right to obtain

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_i \beta_i(\alpha)\frac{\partial}{\partial\alpha_i}\right]\mathbf{F}+\mathbf{Z}_f^{-1}\left(\mu\frac{d}{d\mu}\mathbf{Z}_f\right)\mathbf{F} +\mathbf{F}\left(\mu\frac{d}{d\mu}\tilde{\mathbf{Z}}\right)\tilde{\mathbf{Z}}^{-1}=0, \tag{7.29}$$

where α_i denotes the parameters of the theory (g_R,ξ_R,λ_R), and β_i denotes the corresponding RG function $\beta_i(\alpha):=\mu(\partial/\partial\mu)\alpha_i$. Here we have used a fact that $\mu(\partial/\partial\mu)+\Sigma_i\beta_i[\alpha(\mu)](\partial/\partial\alpha_i)$ is just the ordinary differential operator $\mu(d/d\mu)$.

Defining the RG function (matrix) γ_{f}, $\tilde{\gamma}$ from $\mathbf{Z}_f$, $\tilde{\mathbf{Z}}$ by

$$\mu\frac{d}{d\mu}\mathbf{Z}_f:=\mathbf{Z}_f\gamma_{\mathrm{f}},\quad \mu\frac{d}{d\mu}\tilde{\mathbf{Z}}:=\tilde{\gamma}\tilde{\mathbf{Z}}, \tag{7.30}$$

we obtain the RG equation for the matrix $\mathbf{F}$ of the Wilson coefficients

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_i \beta_i(\alpha)\frac{\partial}{\partial\alpha_i}\right]\mathbf{F}(p,\alpha,\mu)+\gamma_{\mathrm{f}}\mathbf{F}(p,\alpha,\mu) +\mathbf{F}(p,\alpha,\mu)\tilde{\gamma}=0. \tag{7.31}$$

Similarly, we can show that $\mathbf{D}_{\mathrm{pert}}$ obeys the RG equation

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_i \beta_i(\alpha)\frac{\partial}{\partial\alpha_i}\right]\mathbf{D}_{\mathrm{pert}}(p,\alpha,\mu)+\gamma_{\mathrm{f}}\mathbf{D}_{\mathrm{pert}}(p,\alpha,\mu)=0. \tag{7.32}$$

2. Solving the RG equation

Now we proceed to solve the RG equation just obtained. A simple dimensional analysis leads to the relation $\mathbf{F}(\kappa p,\alpha,\kappa\mu)=\kappa^{d_f}\mathbf{F}(p,\alpha,\mu)$ which is equivalent to the relation

$$\mathbf{F}(\kappa p,\alpha,\mu)=\kappa^{d_f}\mathbf{F}\left(p,\alpha,\frac{\mu}{\kappa}\right), \tag{7.33}$$

where d_f is the canonical dimension of $\mathbf{F}$. Hence, $\mathbf{F}$ satisfies

$$\left[\kappa\frac{\partial}{\partial\kappa}+\mu\frac{\partial}{\partial\mu}-d_F\right]\mathbf{F}(\kappa p,\alpha,\mu)=0. \tag{7.34}$$

We use this equation to eliminate $\mu(\partial/\partial\mu)$ in Eq. (7.31) to obtain

$$\left[\kappa\frac{\partial}{\partial\kappa}-\sum_i \beta_i(\alpha)\frac{\partial}{\partial\alpha_i}-d_F\right]\mathbf{F}(\kappa p,\alpha,\mu)-\gamma_{\mathrm{f}}\mathbf{F}(\kappa p,\alpha,\mu) -\mathbf{F}(\kappa p,\alpha,\mu)\tilde{\gamma}=0. \tag{7.35}$$

This is the homogeneous RG equation of Weinberg–'t Hooft type [52] which is adequate for the mass-independent renormalization method.

By the standard method [30,32], the general solution of the RG equation (7.35) is given by

$$\mathbf{F}(\kappa p,\alpha,\mu)=\kappa^{-4}\exp\left\{\int_1^\kappa d\kappa'\frac{\gamma_{\mathrm{f}}(\kappa')}{\kappa'}\right\} \times\mathbf{F}(p,\bar{\alpha}(\kappa),\mu)\exp\left\{\int_1^\kappa d\kappa'\frac{\tilde{\gamma}(\kappa')}{\kappa'}\right\}, \tag{7.36}$$

where we have imposed the boundary condition $\bar{\alpha}(\kappa=1)=\alpha$.

[9]Were it not for the renormalization of the composite operator $\mathbf{F}_0$ reduced to $\mathbf{F}$.

A similar consideration yields the general solution of the RG equation (7.32):

$$\mathbf{D}_{\mathrm{pert}}(\kappa p,\alpha,\mu)=\kappa^{-2}\exp\left\{\int_1^\kappa d\kappa'\frac{\gamma_{\mathrm{f}}(\kappa')}{\kappa'}\right\}\mathbf{D}_{\mathrm{pert}}(p,\alpha,\mu). \tag{7.37}$$

Once we know the Z factors of the fundamental field and the composite operator, it is easy to calculate γ_{f}, $\tilde{\gamma}$ according to Eq. (7.30). If the integrations in the arguments of the exponential in Eqs. (7.36) and (7.37) are performed, the κ dependence of the solution will be exactly determined. However, Z factors are obtained in terms of renormalized parameters g_R, ξ_R, λ_R and hence depend implicitly on κ through them. This fact makes the analysis more difficult in general.

3. Solution around the UV fixed point B

We can calculate γ_{f}, $\tilde{\gamma}$ up to $O(\hbar)$, since we know all the Z factors of the fundamental field and the composite operator up to $O(\hbar)$. In the high-energy limit $\kappa\to\infty$, it is expected that the solution can be explicitly obtained in the neighborhood of the nontrivial UV stable fixed point in the parameter space, due to the asymptotic freedom of Yang-Mills theory, i.e., $\overline{g}(\kappa)\to\overline{g}_\infty=0$ as $\kappa\to\infty$.

In three-dimensional parameter space g_R, ξ_R, λ_R, we have found that all the points are flowing into the UV fixed point B in the UV limit except for some lines that have higher symmetry. On the other hand, within perturbation theory using dimensional regularization, the μ dependent loop correction of all Z factors always appears with a factor of $O(g_R^2)$. Therefore, the RG function γ as an element of the matrix γ defined by differentiating the Z factor with respect to μ is accompanied by g_R^2 to the $O(\hbar)$, similar to $\gamma\sim g_R^2 f(\xi,\lambda)\hbar+O(\hbar^2)$. If the polynomial function $f(\xi,\lambda)$ in the above expression has a nonvanishing value at the fixed point (ξ^*,λ^*), the μ dependence of $\gamma=g^2 f$ is governed by g^2 alone and hence we can replace $f(\xi,\lambda)$ with the constant $f(\xi^*,\lambda^*)$ at the UV fixed point. By substituting the fixed-point values $\lambda_R^*=26/3$, $\xi_R^*=1/2$ into ξ, λ, the Z factors become

$$Z_A^*=1-\frac{13}{6}\frac{g^2N_c}{16\pi^2}\frac{\mu^{-2\epsilon}}{\epsilon},\quad Z_C^*=1-\frac{17}{12}\frac{g^2N_c}{16\pi^2}\frac{\mu^{-2\epsilon}}{\epsilon},$$

$$Z_1^*=1-\frac{29}{4}\frac{g^2N_c}{16\pi^2}\frac{\mu^{-2\epsilon}}{\epsilon},\quad Z_2^*=-\frac{1}{4}\left(\frac{26}{3}\right)^2\frac{g^2N_c}{16\pi^2}\frac{\mu^{-2\epsilon}}{\epsilon},$$

$$Z_3^*=\frac{1}{2}\frac{g^2N_c}{16\pi^2}\frac{\mu^{-2\epsilon}}{\epsilon},\quad Z_4^*=1, \tag{7.38}$$

which yield the matrix of the renormalization group function

$$\gamma_f^*(g)=\frac{g^2N_c}{8\pi^2}\begin{pmatrix}\frac{13}{6} & 0\\ 0 & \frac{17}{12}\end{pmatrix},$$

$$\tilde{\gamma}^*(g)=\frac{g^2N_c}{8\pi^2}\begin{pmatrix}\frac{61}{12} & \frac{1}{4}\left(\frac{26}{3}\right)^2\\ -\frac{1}{2} & -\frac{17}{12}\end{pmatrix}. \tag{7.39}$$

Furthermore, we define the coefficient matrix $\mathbf{C}_{\gamma f}$ and $\mathbf{C}_{\tilde{\gamma}}$ in Eq. (7.39) by

$$\gamma_f^*(g):=g^2\mathbf{C}_{\gamma_f},\quad \tilde{\gamma}^*(g):=g^2\mathbf{C}_{\tilde{\gamma}}. \tag{7.40}$$

By taking into account the RG equation $\mu(d/d\mu)g=-(b/8\pi^2)g^3$ $(b=11/6N_c)$ and the resulting relation $(d/d\mu)\ln g^2=(2/g)/(d/d\mu)g=-(2b/8\pi^2)(g^2/\mu)$, the nontrivial integration of Eq. (7.36) can be performed as

$$\int_1^\kappa d\kappa'\frac{\gamma(\overline{g}(\kappa'))}{\kappa'}=\int_1^\kappa d\kappa'\mathbf{C}_\gamma\frac{[\overline{g}(\kappa')]^2}{\kappa'}=\mathbf{C}_\gamma\frac{8\pi^2}{2b}\ln\frac{\overline{g}^2(1)}{\overline{g}^2(\kappa)}. \tag{7.41}$$

Hence the solution becomes

$$\mathbf{F}(\kappa p,\alpha,\mu)=\kappa^{-4}\left(\frac{\overline{g}^2(1)}{\overline{g}^2(\kappa)}\right)^{\mathbf{C}_{\gamma_{\mathrm{f}}}(8\pi^2/2b)}\{\mathbf{F}(p,\overline{\alpha}(\kappa),\mu)\}\times\left(\frac{\overline{g}^2(1)}{\overline{g}^2(\kappa)}\right)^{\mathbf{C}_{\tilde{\gamma}}(8\pi^2/2b)}. \tag{7.42}$$

The κ dependence of $\overline{g}^2$ is obtained by solving its RG equation as $\overline{g}^2(\kappa)\sim[(2b/8\pi^2)\ln\kappa]^{-1}$ for large κ. Substituting Eqs. (7.41) into (7.36), therefore, we determine the $\ln\kappa$ dependence of the solution for large κ

$$\mathbf{F}(\kappa p,\alpha,\mu)=\kappa^{-4}(\ln\kappa)^{\mathbf{C}_{\gamma_f}(8\pi^2/2b)}\{\mathbf{F}(p,\overline{\alpha}(\kappa),\mu)\}\times(\ln\kappa)^{\mathbf{C}_{\tilde{\gamma}}(8\pi^2/2b)}. \tag{7.43}$$

In order to cast the matrix power of $\ln\kappa$ into a more tractable form, we shall diagonalize the matrix $\mathbf{C}_{\tilde{\gamma}}$ in such a way that $\mathbf{S}$ diagonalizes $\mathbf{C}_{\tilde{\gamma}}$ by the similarity transformation $\mathbf{C}_{\tilde{\gamma}}\to\mathbf{S}^{-1}\cdot\mathbf{C}_{\tilde{\gamma}}\cdot\mathbf{S}$. Such a matrix $\mathbf{S}$ and the diagonalized matrix are given by

$$\mathbf{S}=\begin{pmatrix}-\frac{13}{3} & -\frac{26}{3}\\ 1 & 1\end{pmatrix},\quad \mathbf{S}^{-1}\cdot\mathbf{C}_{\tilde{\gamma}}\cdot\mathbf{S}=\frac{N_c}{8\pi^2}\begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{35}{12}\end{pmatrix}. \tag{7.44}$$

This diagonalization corresponds to redefining the combination between two composite operators of mass dimension 2, i.e., $1/2A(0)A(0)$ and $i\overline{C}(0)C(0)$, by multiplying $\mathbf{S}^{-1}$:

$$\begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \end{pmatrix} = S^{-1} \begin{pmatrix} \frac{1}{2}A^2 \\ i\bar{C}C \end{pmatrix} = \frac{3}{13} \begin{pmatrix} \frac{1}{2}A^2 + \frac{26}{3} i\bar{C}C \\ -\frac{1}{2}A^2 - \frac{13}{3} i\bar{C}C \end{pmatrix}. \tag{7.45}$$

Inserting the identity matrix $\mathbf{1} = \mathbf{S}\cdot\mathbf{S}^{-1}$ appropriately, the solution (7.42) is rewritten as

$$\mathbf{F}(\kappa p,\alpha,\mu) = \kappa^{-4} \left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{\mathbf{C}_{\gamma_f}(8\pi^2/2b)} \mathbf{F}(p,\bar{\alpha},\mu) \times \mathbf{S}\cdot\mathbf{S}^{-1} \left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{\mathbf{C}_{\tilde{\gamma}}(8\pi^2/2b)} \mathbf{S}\cdot\mathbf{S}^{-1}. \tag{7.46}$$

Now both $\mathbf{C}_{\gamma_f}$ and $\mathbf{S}^{-1}\cdot\mathbf{C}_{\tilde{\gamma}}\cdot\mathbf{S}$ are diagonal. Hence we can write down the power explicitly as

$$\mathbf{F}(\kappa p) = \kappa^{-4} \begin{pmatrix} \left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{(13/6)(N_c/2b)} & 0 \\ 0 & \left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{(17/12)(N_c/2b)} \end{pmatrix} \mathbf{T}(p)\mathbf{S} \begin{pmatrix} \left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{(3/4)(N_c/2b)} & 0 \\ 0 & \left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{(35/12)(N_c/2b)} \end{pmatrix} \mathbf{S}^{-1}. \tag{7.47}$$

Here we impose a condition that $\mathbf{T}(p) \coloneqq \mathbf{F}(p,\bar{\alpha}(\kappa),\mu)$ coincides with the Wilson coefficient in the tree level obtained in the previous section in which the coupling constant is replaced with the running coupling constant $\bar{\alpha}(\kappa)$. Note that $\mathbf{F}$ is the Wilson coefficient of the Green function [not of the one-particle irreducible (1PI) function].[10] Hence we put

$$\mathbf{T}(p) = \begin{pmatrix} T_1(p) & T_2(p) \\ T_3(p) & T_4(p) \end{pmatrix} = \frac{N_c \bar{g}^2(\kappa)}{2(N_c^2-1)} \begin{pmatrix} -(iD_0)^2(1+\lambda)P_T & (iD_0)^2 4\xi(1-\xi)P_L \\ (iG_0)^2 & 0 \end{pmatrix}. \tag{7.48}$$

We notice that each element $T_1,\ldots,T_4$ of $\mathbf{T}(p)$ brings an extra $\ln\kappa$ factor to $\mathbf{F}$ through $\bar{g}^2(\kappa) \sim 1/\ln\kappa$. Therefore, the OPE correction up to dimension 2 operators reads

$$\mathbf{F}(p)\begin{pmatrix} \frac{1}{2}A^2 \\ i\bar{C}C \end{pmatrix} = \begin{pmatrix} \left(-\frac{13}{3}T_1 + T_2\right)\left(\frac{\ln p/\Lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}\right)^{(35/12)\,(N_c/2b)} & \left(-\frac{26}{3}T_1 + T_2\right)\left(\frac{\ln p/\Lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}\right)^{(61/12)\,(N_c/2b)} \\ -\frac{13}{3}T_3\left(\frac{\ln p/\Lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}\right)^{(13/6)\,(N_c/2b)} & -\frac{26}{3}T_3\left(\frac{\ln p/\Lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}\right)^{(13/3)\,(N_c/2b)} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \end{pmatrix}, \tag{7.49}$$

where we have used $T_4 = 0$. Here we have used the translation rule from the MS scheme to the MOM scheme

$$\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)} \to \frac{\ln p/\Lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}. \tag{7.50}$$

Among the terms with various powers of $\ln\kappa$, the largest-power term (corresponding to the largest eigenvalue of the matrix $\mathbf{C}_\gamma$) is dominant in the UV limit $(\kappa \gg 1)$. Extracting this $\ln\kappa$ contribution, we can simplify the Wilson coefficient of the 1PI function in the UV limit as

$$\mathbf{C}^{\mathrm{1PI}} = \begin{pmatrix} C_{gl}^{[A^2]} & C_{gl}^{[\bar{C}C]} \\ C_{gh}^{[A^2]} & C_{gh}^{[\bar{C}C]} \end{pmatrix} = \begin{pmatrix} (iD_{\mathrm{pert}})^{-2} & 0 \\ 0 & (iG_{\mathrm{pert}})^{-2} \end{pmatrix} \mathbf{F} \tag{7.51}$$

[10]Except for the Landau gauge in which no operator mixing occurs, a linear combination of different powers of $\ln\kappa$ appears in the solution, and its combination coefficients cannot be completely determined by perturbation theory alone. But it is important to note that a fitting of the analytical result with the simulation data (or experimental data) can determine the asymptotic behavior of $\mathbf{F}$ completely as discussed in the next subsection.

$$
= \frac{8\pi^2}{2b}\frac{N_c}{(N_c^2-1)}\begin{pmatrix} (D_{\text{pert}}/D_0)^{-2} & 0 \\ 0 & (G_{\text{pert}}/G_0)^{-2}\end{pmatrix}
$$
$$
\times\begin{pmatrix} \frac{13(-1-\lambda)P_T-6\xi(1-\xi)P_L}{13}\frac{\left(\ln\frac{p}{\Lambda_{\text{QCD}}}\right)^{-1+(61/12)(N_c/2b)}}{\left(\ln\frac{\mu}{\Lambda_{\text{QCD}}}\right)^{(61/12)(N_c/2b)}} & \frac{13(-1-\lambda)P_T-6\xi(1-\xi)P_L}{3}\frac{\left(\ln\frac{p}{\Lambda_{\text{QCD}}}\right)^{(-1+(61/12)(N_c/2b)}}{\left(\ln\frac{\mu}{\Lambda_{\text{QCD}}}\right)^{(61/12)(N_c/2b)}} \\ -1\frac{\left(\ln\frac{p}{\Lambda_{\text{QCD}}}\right)^{(-1+(13/3)(N_c/2b)}}{\left(\ln\frac{\mu}{\Lambda_{\text{QCD}}}\right)^{(26/6)(N_c/2b)}} & -\frac{13}{3}\frac{\left(\ln\frac{p}{\Lambda_{\text{QCD}}}\right)^{-1+(13/3)(N_c/2b)}}{\left(\ln\frac{\mu}{\Lambda_{\text{QCD}}}\right)^{(26/6)(N_c/2b)}}\end{pmatrix}. \tag{7.52}
$$

In the similar way, we obtain

$$
\mathbf{D}_{\text{pert}}(\kappa p,\alpha,\mu)=\kappa^{-2}\begin{pmatrix}\left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{(13/6)(N_c/2b)} & 0 \\ 0 & \left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{(17/12)(N_c/2b)}\end{pmatrix}\mathbf{D}_t(p), \tag{7.53}
$$

where the tree expression is given by

$$
\mathbf{D}_t(p)=\begin{pmatrix} D_0(p) \\ -iG_0(p)\end{pmatrix}=\begin{pmatrix} -\frac{1}{p^2}(P_T+\lambda P_L) \\ \frac{1}{p^2}\end{pmatrix}. \tag{7.54}
$$

4. The solution at the conventional Landau gauge

Finally, we consider the OPE on line A of the fixed points (corresponding to the conventional Landau gauge), the RG matrices read

$$
\gamma_f^* = g^2 C_{\gamma_f} = \frac{g^2 N_c}{8\pi^2}\begin{pmatrix} -\frac{13}{6} & 0 \\ 0 & -\frac{3}{4}\end{pmatrix}, \quad \tilde{\gamma}^* = g^2 C_{\tilde{\gamma}} = \frac{g^2 N_c}{8\pi^2}\begin{pmatrix} -\frac{35}{12} & 0 \\ -\frac{1}{2} & \frac{3}{4}\end{pmatrix}. \tag{7.55}
$$

The diagonalization can be performed as

$$
\mathbf{S}=\begin{pmatrix} 0 & -\frac{13}{3} \\ 1 & 1\end{pmatrix}, \quad \mathbf{S}^{-1}\cdot\mathbf{C}_{\tilde{\gamma}}\cdot\mathbf{S}=\frac{N_c}{8\pi^2}\begin{pmatrix}\frac{3}{4} & 0 \\ 0 & \frac{35}{12}\end{pmatrix}. \tag{7.56}
$$

The eigenvalues of $\mathbf{C}_{\tilde{\gamma}}$ are the same as those at fixed point B. Therefore, we obtain the Wilson coefficient $C_{\mu\nu}^{[A^2]}$ between $\langle A_\mu(p)A_\nu(-p)\rangle^{-1}$ and $\langle (A(0))^2\rangle$ and $C^{[C\bar{C}]}$ between $\langle C(p)\bar{C}(-p)\rangle^{-1}$ and $\langle (A(0))^2\rangle$:

$$
\mathbf{F}(\kappa p)=\kappa^{-4}\begin{pmatrix} T_1(p)\left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{(3/4)(N_c/2b)} & 0 \\ T_3(p)\left(\frac{\bar{g}^2(1)}{\bar{g}^2(\kappa)}\right)^{(13/6)(N_c/2b)} & 0\end{pmatrix}, \tag{7.57}
$$

where no mixing between gluon and ghost occurs due to $T_2=0$ in addition to $T_4=0$. The coefficients of the 1PI OPEs read

$$\begin{pmatrix} C_{gl}^{[A^2]} & C_{gl}^{[\bar{C}C]} \\ C_{gh}^{[A^2]} & C_{gh}^{[\bar{C}C]} \end{pmatrix} = \frac{8\pi^2}{2b}\frac{N_c}{2(N_c^2-1)} \begin{pmatrix} -\left(\frac{D_{\rm pert}}{D_0}\right)^{-2} \frac{\left(\ln\frac{p}{\Lambda_{\rm QCD}}\right)^{(3/4)(N_c/2b)-1}}{\left(\ln\frac{\mu}{\Lambda_{\rm QCD}}\right)^{(3/4)(N_c/2b)}} & 0 \\ \left(\frac{G_{\rm pert}}{G_0}\right)^{-2} \frac{\left(\ln\frac{p}{\Lambda_{\rm QCD}}\right)^{(13/6)(N_c/2b)-1}}{\left(\ln\frac{\mu}{\Lambda_{\rm QCD}}\right)^{(13/6)(N_c/2b)}} & 0 \end{pmatrix}, \tag{7.58}$$

where

$$\mathbf{D}_{\rm pert}(p) = \begin{pmatrix} \left(\frac{\ln p/\Lambda_{\rm QCD}}{\ln \mu/\Lambda_{\rm QCD}}\right)^{-(13/6)(N_c/2b)} & 0 \\ 0 & \left(\frac{\ln p/\Lambda_{\rm QCD}}{\ln \mu/\Lambda_{\rm QCD}}\right)^{-(3/4)(N_c/2b)} \end{pmatrix} \mathbf{D}_t(p). \tag{7.59}$$

This result for the ghost part is new, while the gluon part reproduces the recent result of Boucaud *et al.* [22] in the MOM scheme. (Note that their definition of γ is different from ours by a factor 2 and the coefficient γ_0 differs by the signature.) In order to transfer from our renormalization scheme to the MOM scheme, we have used the translation rule (7.50). In the Landau gauge, therefore, we have confirmed that the ghost condensation does not affect the inverse gluon propagator as in the tree level, even if the leading logarithmic corrections are taken into account in the OPE. In other words, the gluon condensation is decoupled from the ghost condensation within this approximation.

C. Full propagators: Momentum dependence

The vacuum polarization tensor of the gluon is decomposed into transverse and longitudinal parts

$$\Pi_{\mu\nu}^{AB}(p) = [\Pi^T(p^2)P_{\mu\nu}^T + \Pi^L(p^2)P_{\mu\nu}^L]\delta^{AB}, \tag{7.60}$$

where Π^T and Π^L are functions of p^2 alone. Once the vacuum polarization functions Π^T and Π^L of the gluon are obtained from the OPE, the propagator is written as

$$(D)_{\mu\nu}^{AB}(p) = \delta^{AB}\left[\frac{1}{-p^2+\Pi^T(p^2)}P_{\mu\nu}^T + \frac{\lambda}{-p^2+\lambda\Pi^L(p^2)}P_{\mu\nu}^L\right] \tag{7.61}$$

$$= \delta^{AB}\left[\frac{Z_{gl}(-p^2)}{-p^2}P_{\mu\nu}^T + \frac{\lambda}{-p^2+\lambda\Pi^L(p^2)}P_{\mu\nu}^L\right], \tag{7.62}$$

where we have defined a function $Z_{gl}(-p^2)$ by

$$Z_{gl}(-p^2) = Z_{\rm pert}(-p^2) + Z_{\rm OPE}(-p^2) := \frac{-p^2}{-p^2+\Pi^T(p^2)}. \tag{7.63}$$

Note that $\Pi^L(p^2) \equiv 0$ in the conventional Landau gauge.

On the other hand, if the vacuum polarization function of the ghost $\Pi_{gh}^{AB}(p^2) = \delta^{AB}\Pi_{gh}(p^2)$ is calculated by the OPE, the ghost propagator is obtained as

$$G^{AB}(p) = [(G_0)^{-1} + i\Pi_{gh}(p^2)]_{AB}^{-1} = \frac{1}{-ip^2+i\Pi_{gh}(p^2)}\delta^{AB} = (-i)\frac{G_{gh}(-p^2)}{-p^2}\delta^{AB}, \tag{7.64}$$

where we have introduced a function $G_{gh}(-p^2)$ by

$$G_{gh}(-p^2) = G_{\rm pert}(-p^2) + G_{\rm OPE}(-p^2) := \frac{-p^2}{-p^2+\Pi_{gh}(p^2)}. \tag{7.65}$$

The OPE contribution $\Pi^{\rm OPE}$ to the vacuum polarization function in the inverse propagators (7.3) and (7.12) is related to the Wilson coefficient $\mathbf{C}^{\rm 1PI}$ as

$$\Pi^{\mathrm{OPE}} := \begin{pmatrix} \Pi_{gl}^{\mathrm{OPE}} \\ \Pi_{gh}^{\mathrm{OPE}} \end{pmatrix} = \mathbf{C}^{\mathrm{PI}} \begin{pmatrix} \frac{1}{2}A^2 \\ i\bar{C}C \end{pmatrix} = \begin{pmatrix} (iD_{\mathrm{pert}})^{-2} & 0 \\ 0 & (iG_{\mathrm{pert}})^{-2} \end{pmatrix} \mathbf{F} \begin{pmatrix} \frac{1}{2}A^2 \\ i\bar{C}C \end{pmatrix}. \tag{7.66}$$

Substituting the result (7.49) into Eq. (7.66), we obtain a pair of vacuum polarization functions

$$\Pi^{\mathrm{OPE}}(p) = \begin{pmatrix} \left(T_2 - \frac{13}{3}T_1\right)\left(\frac{\ln p/\lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}\right)^{(35/12)(N_c/2b)} \frac{1}{(iD_{\mathrm{pert}})^2} & \left(T_2 - \frac{26}{3}T_1\right)\left(\frac{\ln p/\Lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}\right)^{(61/12)(N_c/2b)} \frac{1}{(iD_{\mathrm{pert}})^2} \\ -\frac{13}{3}T_3\left(\frac{\ln p/\Lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}\right)^{(13/6)(N_c/2b)} \frac{1}{(iG_{\mathrm{pert}})^2} & -\frac{26}{3}T_3\left(\frac{\ln p/\Lambda_{\mathrm{QCD}}}{\ln \mu/\Lambda_{\mathrm{QCD}}}\right)^{(13/3)(N_c/2b)} \frac{1}{(iG_{\mathrm{pert}})^2} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \end{pmatrix}. \tag{7.67}$$

It turns out that the vacuum polarization functions just obtained reduce to the tree results, i.e., Eqs. (7.10) and (7.19), at $\kappa = 1$ (or $p = \mu$). Therefore, the ghost condensation $\langle i\bar{C}C\rangle$ contributes to the gluon and ghost vacuum polarization functions in the leading logarithmic corrections of the OPE.

Thus the following OPE contribution to the gluon and ghost vacuum polarization functions are obtained:

$$\Pi_T^{\mathrm{OPE}}(p^2) = \frac{2\pi^2}{b}\frac{N_c(1+\lambda)}{(N_c^2-1)} \left\{ \frac{\left(\ln\frac{p}{\Lambda_{\mathrm{QCD}}}\right)^{(35/12)(N_c/2b)-1}}{\left(\ln\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)^{(35/12)(N_c/2b)}} \left(\left\langle\frac{1}{2}A^2\right\rangle + \frac{26}{3}\langle i\bar{C}C\rangle\right) - 2\frac{\left(\ln\frac{p}{\Lambda_{\mathrm{QCD}}}\right)^{(61/12)(N_c/2b)-1}}{\left(\ln\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)^{(61/12)(N_c/2b)}} \left(\left\langle\frac{1}{2}A^2\right\rangle + \frac{13}{3}\langle i\bar{C}C\rangle\right) \right\} \left(\frac{D_0(p)}{D_{\mathrm{pert}}(p)}\right)^2, \tag{7.68}$$

$$\Pi_{\mathrm{gh}}^{\mathrm{OPE}}(p^2) = \frac{2\pi^2}{b}\frac{N_c}{(N_c^2-1)} \left\{ -\frac{\left(\ln\frac{p}{\Lambda_{\mathrm{QCD}}}\right)^{(13/6)(N_c/2b)-1}}{\left(\ln\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)^{(13/6)(N_c/2b)}} \left(\left\langle\frac{1}{2}A^2\right\rangle + \frac{26}{3}\langle i\bar{C}C\rangle\right) + 2\frac{\left(\ln\frac{p}{\Lambda_{\mathrm{QCD}}}\right)^{(13/3)(N_c/2b)-1}}{\left(\ln\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)^{(13/3)(N_c/2b)}} \left(\left\langle\frac{1}{2}A^2\right\rangle + \frac{13}{3}\langle i\bar{C}C\rangle\right) \right\} \left(\frac{G_0(p)}{G_{\mathrm{pert}}(p)}\right)^2. \tag{7.69}$$

The effective gluon mass is obtained from the pole of $Z_{gl}(-p^2)$, i.e., a solution of the equation $p^2 = \Pi_T(p^2)$, while the effective ghost mass is obtained from the pole of $G_{gh}(-p^2)$, i.e., a solution of the equation $p^2 = -i\Pi_{gh}(p^2)$. In view of this, the solutions (7.68) and (7.69) would give an improvement of the tree-level results (7.11) and (7.20). However, a BRST non-invariant combination $\mathcal{Q}_2$ of composite operators appears together with the BRST invariant combination $\mathcal{Q}_1$ discussed in the previous section. Therefore, these results indicate that we need more endeavor in order to reach the BRST invariant pole position in the IR region.

In the Landau gauge, especially, we have

$$Z_{gl}(-p^2) = -p^2 D_{\mathrm{pert}}(p) - p^{-2}\left(\frac{\pi^2}{b}\frac{N_c}{(N_c^2-1)} \frac{\left(\ln\frac{p}{\Lambda_{\mathrm{QCD}}}\right)^{(3/4)(N_c/2b)-1}}{\left(\ln\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)^{(3/4)(N_c/2b)}} \langle A^2\rangle \right), \tag{7.70}$$

$$G_{gh}(-p^2) = -ip^2 G_{\mathrm{pert}}(p) + p^{-2}\left(\frac{\pi^2}{b} \frac{N_c}{(N_c^2-1)} \frac{\left(\ln \frac{p}{\Lambda_{\mathrm{QCD}}}\right)^{(13/6)(N_c/2b)-1}}{\left(\ln \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)^{(13/6)(N_c/2b)}} \langle A^2 \rangle \right). \tag{7.71}$$

After the Wick rotation to the Euclidean region $p^2 \to -p_E^2$, we find that the function $Z_{gl}(p_E^2)$ is monotonically increasing in p_E^2 if $\langle A_E^2 \rangle := -\langle A^2 \rangle > 0$, as in the case of constant $\Pi^T(-p_E^2) = M^2 > 0$. On the other hand, if $\langle A_E^2 \rangle := -\langle A^2 \rangle < 0$, $Z_{gl}(p_E^2)$ has a Landau pole in the IR region and is monotonically decreasing in p_E^2 in the UV region. In the conventional Landau gauge, these results can be compared with those of the Schwinger-Dyson equation (see, e.g., Ref. [53]) and the numerical simulation on a lattice (see, e.g., Refs. [22, 54–56]). According to these results, $Z_{gl}(p_E^2)$ is enhanced at intermediate momenta and has a peak at about 1 GeV. It was argued [56] that the enhancement of the gluonic form factor at IR region is related to quark confinement. However, this region is beyond the reach of the present study. Incidentally, data in a gauge other than the Landau gauge is not yet available.

VIII. CONCLUSION AND DISCUSSION

In this paper we have discussed the multiplicative renormalizability of the composite operator $\mathcal{O}$ in QED and Yang-Mills theory. This research is motivated by clarifying the mechanism of mass generation and a possible connection to quark confinement.

In QED, we have shown that the composite operator is trivially renormalizable and that the renormalized composite operator is BRST and anti-BRST invariant for an arbitrary value of the gauge fixing parameter. There is no subtlety related to the renormalization of the composite operator.

In Yang-Mills theory, we have adopted the most general Lorentz gauge with two gauge-fixing parameters ξ, λ which was derived by Baulieu and Thierry-Mieg [8]. We knew [2] that the bare composite operator $\mathcal{O}$ of mass dimension 2 is invariant under the *bare* BRST and anti-BRST transformations for the choice of gauge parameters $\lambda = 0$ or $\xi = 1/2$ and that it is also invariant under the gauge transformation in the Landau gauge $\lambda = 0$. In this paper the composite operator has been renormalized by taking into account the operator mixing carefully. Here the matrix of renormalization factors has been explicitly calculated. Consequently, we have found that the BRST and anti-BRST invariance of the renormalized composite operator $\mathcal{O}^R$ holds if the renormalized parameters take the same value, $\lambda_R = 0$ or $\xi_R = 1/2$, as the bare one. Moreover, we have obtained the RG flow in the (ξ, λ) plane to one-loop order. In the RG flow diagram, the RG flow runs only on the line $\xi_R = 1/2$ if the initial position of ξ is located somewhere on the line. The line $\lambda_R = 0$ is a line of fixed points. Therefore, if the system is located on a point in the line $\lambda_R = 0$ initially, it cannot move from the initial position. This fact guarantees the BRST invariance of the renormalized composite operator $\mathcal{O}^R$.

We have also examined how the conventional calculations are modified in the presence of the vacuum condensate of mass dimension 2. By performing the OPE of the gluon and ghost propagators, we have shown that the effective masses of gluon and ghost are generated due to the nonvanishing vacuum condensate. Although this phenomenon was already suggested based on the tree level calculation, we have taken into account the leading logarithmic corrections consistent with the RG flow by making use of the RG equation. We have found that the effective masses are provided from the ghost condensation $\langle i\bar{\mathcal{C}} \cdot \mathcal{C} \rangle$ as well as the gluon condensation $\langle 1/2 \mathcal{A}_\mu \cdot \mathcal{A}^\mu \rangle$ (except for the Landau gauge $\lambda = 0$). This result should be compared with the tree level result where the effective mass has a contribution from the gluon condensate alone.

The next step is to show that the nonvanishing vacuum condensate $\langle \mathcal{O} \rangle \neq 0$ is actually realized in the QCD vacuum. An attempt in this direction has already been performed in Ref. [20] by calculating the effective potential for the ghost condensation $\langle i\bar{C}C \rangle$ in the $SU(2)$ and $SU(3)$ Yang-Mills theories in the MA gauge. Quite recently, Verschelde *et al.* [57] have obtained the multiplicatively renormalizable effective potential for the gluon condensate $\langle 1/2 \mathcal{A}_\mu \mathcal{A}^\mu \rangle$ in the Landau gauge up to two-loop order in the $SU(N)$ Yang-Mills theory. Both results support that the nonzero vacuum condensate of mass dimension 2 is energetically favored in Yang-Mills theory. In these approaches, an auxiliary field $\rho(x)$ corresponding to the composite operator has been introduced to obtain the effective potential $V(\sigma)$ of a constant $\sigma = \rho(x)$. However, this treatment has a number of subtle points which have not been discussed in these papers. This issue will be discussed in a subsequent paper [27] in detail.

In massless QED, photon pairing [43,44] can occur in the strong coupling phase [39–41] where the chiral symmetry is spontaneously broken. Therefore, it will be possible to discuss the interplay between quark confinement and chiral symmetry breaking on equal footing in a unified treatment. The extension of this viewpoint into the non-Abelian case, i.e., gluon pairing [42] is also an interesting subject for future work.

Finally, we point out that the operator $\mathcal{O}$ is essentially a mass term for the gluon and ghost fields. Although a naive introduction of a mass term for the gluon alone breaks the BRST symmetry, our result indicates that there is a BRST invariant combination of mass terms

$$\mathcal{L}_m := \mathrm{tr}\left[\frac{1}{2} m^2 \mathcal{A}_\mu(x) \cdot \mathcal{A}_\mu(x) + m^2 \alpha i \bar{\mathcal{C}}(x) \cdot \mathcal{C}(x) \right]. \tag{8.1}$$

This mass term is very similar to that obtained after the spontaneous breakdown caused by the nonvanishing vacuum expectation value of the Higgs scalar field. In our case, the mass should be of dynamical origin. It is possible to give a proof of the multiplicative renormalizability of the Yang-Mills theory with a mass term preserving the BRST symmetry to all orders of perturbation theory. However, it is known [58,59] that the introduction of the mass term (8.1) breaks the nilpotency of the off-shell BRST transformation as well as the on-shell one. Consequently, the unitarity of the theory turns out to be spoiled. In this sense, the mass generation should occur in a dynamical way, i.e., $\langle \mathcal{O} \rangle \neq 0$ in the limit $m \to 0$. This viewpoint will be discussed in a subsequent paper.

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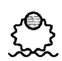
APPENDIX: OPE CALCULATIONS

In order to give the OPE correction for the gluon propagator, we need to calculate the following diagrams:

$$
\begin{aligned}
&= \frac{1}{(N_c^2-1)D} g^{\rho\rho'} \delta^{CC'} g f^{ACD} [g_{\mu\sigma}(-2p)_\rho + g_{\sigma\rho} p_\mu + g_{\rho\mu} p_\sigma] \\
&\quad \times \frac{-i}{p^2} (P^{\mathrm{T}}_{\sigma\sigma'} + \lambda P^{\mathrm{L}}_{\sigma\sigma'}) \delta^{DD'} g f^{BD'C'} [g_{\nu\sigma'}(-2p)_{\rho'} + g_{\sigma'\rho'} p_\nu + g_{\rho'\nu} p_{\sigma'}] \\
&= \frac{1}{(N_c^2-1)D} i g^2 (-N_c) \frac{1}{p^2} [(4+\lambda) p^2 P^{\mathrm{T}}_{\mu\nu} + (D-1) p^2 P^{\mathrm{L}}_{\mu\nu}] \delta^{AB}.
\end{aligned}
\tag{A1}
$$

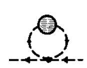


$$
\begin{aligned}
&= \frac{1}{(N_c^2-1)D} i g^2 g^{\gamma\delta} \delta^{CD} \Big[f^{EAB} f^{ECD} (g_{\mu\gamma} g_{\nu\delta} - g_{\mu\delta} g_{\nu\gamma}) \\
&\quad + f^{EAC} f^{EBD} (g_{\mu\nu} g_{\gamma\delta} - g_{\mu\delta} g_{\gamma\nu}) + f^{EAD} f^{EBC} (g_{\mu\nu} g_{\gamma\delta} - g_{\mu\gamma} g_{\delta\nu}) \Big] \\
&= i g^2 \frac{2}{(N_c^2-1)D} N_c [g_{\mu\nu}(D-1)] \delta^{AB}.
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
&= \frac{1}{(N_c^2-1)} (p_\mu + \xi(-p)_\mu) g f^{ADC} \frac{-1}{p^2} \delta^{CC'} (0 + \xi p_\nu) g f^{BC'D'} \delta^{DD'} \\
&= \frac{N_c g^2}{(N_c^2-1)} \frac{1}{p^2} \xi(1-\xi) p_\mu p_\nu \delta^{AB}.
\end{aligned}
\tag{A3}
$$

For the correction of the ghost propagator, we need the calculation of the following diagrams:

$$
\begin{aligned}
&= \frac{\delta^{CC'} g_{\mu\nu}}{(N_c^2-1)D} i g f^{CAD} (p_\mu) \frac{-1}{p^2} \delta^{DD'} i g f^{C'D'B} (p_\nu) \\
&= -\frac{N_c g^2}{(N_c^2-1)D} \delta^{AB} .
\end{aligned}
\tag{A4}
$$



$$
\begin{aligned}
&= -\frac{\delta^{CD}}{(N_c^2-1)} (-i g^2) \lambda \xi (1-\xi) (f^{EAB} f^{EDC} - f^{EAC} f^{EDB}) \\
&= i \frac{N_c g^2}{(N_c^2-1)} \lambda \xi (1-\xi) \delta^{AB}.
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
&= \frac{\delta^{DD'}}{(N_c^2-1)} i g f^{DAC} (-p_\mu)(1-\xi) i \frac{-1}{p^2} (P_{\mathrm{T}} + \lambda P_{\mathrm{L}})^{\mu\nu} \delta^{CC'} i g f^{D'C'B} (\xi(-p_\nu)) \\
&= -i \frac{N_c g^2}{(N_c^2-1)} \lambda \xi (1-\xi) \delta^{AB} .
\end{aligned}
\tag{A6}
$$

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